DYNAMICS OF CONTINUED FRACTIONS AND DISTRIBUTION OF MODULAR SYMBOLS

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Abstract. We formulate a thermodynamical approach to the study of distribution of modular symbols, motivated by the work of Baladi-Vallée. We introduce the modular partitions of continued fractions and observe that the statistics for modular symbols follow from the behavior of modular partitions. We prove the limit Gaussian distribution and residual equidistribution for modular partitions as a vector-valued random variable on the set of rationals whose denominators are up to a fixed positive integer by studying the spectral properties of transfer operator associated to the underlying dynamics. The approach leads to a few applications. We show an average version of conjectures of Mazur-Rubin on statistics for the period integrals of an elliptic newform. We further observe that the equidistribution of mod $p$ values of modular symbols leads to mod $p$ non-vanishing results for special modular $L$-values twisted by a Dirichlet character.

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1. Introduction and statements of results

Non-vanishing of twisted $L$-values seems to genuinely rely on the types of equidistribution/density results of special algebraic cycles. The first prominent example goes back to Ferrero-Washington [12] and Washington [41] for mod $p$ non-vanishing of special Dirichlet $L$-values. A key lemma used in their proof precisely comes from $p$-adic dynamical analogue of the classical density result of Kronecker in ergodic theory. Another view of Ferrero-Washington was later given by Sinnott [35]. Using

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Zariski density of the twisted image of one-parameter formal torus inside a multi-dimensional torus, that is, an algebraic analogue of Kronecker’s theorem, Sinnott presented a simple but exquisite approach.

The approaches have been extensively generalised to the cases of anti-cyclotomic twists. Hida [14] established a geometric generalisation of Sinnott’s method for general CM-points on a modular variety to study mod $p$ non-vanishing of Hecke $L$-values. This was further developed by Burungale-Hida [8] to consider Rankin-Selberg $L$-values. Vatsal [40] also observed the equidistribution of Heegner points on a Gross curve associated to definite quaternion algebras. Meanwhile, up until now, there has been no notable analogous result for the cyclotomic cases except Kim-Sun [18].

The motivation of present research is to explore dynamical counterparts in the study of modular $L$-values with cyclotomic twists or more generally Dirichlet twists. Our early exploration is guided by the fact that the special $L$-values twisted by a Dirichlet character is represented by the period integrals of a cuspform, so-called the modular symbols. Towards non-vanishing result, one may wonder how the symbols are distributed. In this article, we observe that the statistics for the modular symbols is described by Ruelle’s thermodynamic formalism of dynamics of continued fractions.

1.1. Non-vanishing mod $p$ of modular $L$-values. Let $f$ be a newform for $\Gamma_0(N)$ and of weight 2 with the Fourier coefficients $a_f(n)$. Let $\chi$ be a Dirichlet character of conductor $M$. We denote by $L(s, f, \chi)$ the twisted modular $L$-function, which is given as the meromorphic continuation of the Dirichlet series with the coefficients $a_f(n)\chi(n)$. Let $\mathbb{Q}_f$ be the field generated by the coefficients $a_f(n)$ over $\mathbb{Q}$. There are suitable periods $\Omega_f^\pm$ such that the following normalised special $L$-values are algebraic, more precisely,

$$L_f(\chi) := \frac{G(\chi) L(1, f, \chi)}{\Omega_f^\pm} \in \mathbb{Q}_f(\chi)$$

where $G(\chi)$ denotes the Gauss sum and $\pm$ corresponds to the sign $\chi(-1) = \pm 1$. One can choose suitable periods $\Omega_f^\pm$ so that the corresponding algebraic parts $L_f(\chi)$ are $p$-integral with the minimum $p$-adic valuation when, for example, the mod $p$ Galois representation $\rho_{f,p}$ is irreducible, $p$ does not divide $2N$, and $N \geq 3$ (see §1.2 or §3.2).

From now on, let us use these periods when we study mod $p$ equidistribution.

In these circumstances, the $p$-integral $L$-values are expected to be generically non-vanishing modulo $p$. The first non-vanishing result goes back to Ash-Stevens [1] and Stevens [37] for a large class of characters. The cyclotomic case has recently been studied by Kim-Sun [18] who obtain the non-vanishing result for a positive proportion of characters $\chi$ of $\ell$-power conductors with a prime $\ell \neq p$. Their results mainly follow from the generation of the first homology group of the corresponding modular curve by a class of homology cycles that comes from the cyclotomic setting.

We obtain a version of the mod $p$ non-vanishing result which relies on residual equidistribution results on the period integrals, so-called modular symbols,

$$m^+_f(r) := \frac{1}{\Omega_f^+} \left\{ \int_r^{i\infty} f(z)dz \pm \int_{-r}^{i\infty} f(z)dz \right\} \in \mathbb{Q}_f$$
for \( r \in \mathbb{Q} \), by using the following expression for special values

\[
L_f(\chi) = \sum_{a \in (\mathbb{Z}/M\mathbb{Z})^\times} \overline{\chi}(a) \cdot m_f^\pm \left( \frac{a}{M} \right).
\]

Our result from the dynamical setting is the following:

**Theorem A** (Corollary 3.8). Let \( N \geq 3 \) and \( p \nmid 2N \). Let \( f \) be an elliptic newform such that \( \rho_{f,p} \) is irreducible. Then, we have

\[
\# \left\{ \chi \in \hat{\left( \mathbb{Z}/n\mathbb{Z} \right)} \times \mid n \leq M, L_f(\chi) \not\equiv 0 \pmod{p^{1+v_p(\phi(n))}} \right\} \gg M
\]

where \( p \) is a prime over \( p \) in \( \overline{\mathbb{Q}}_p \) and \( v_p(\phi(n)) \) is the \( p \)-adic valuation of the Euler totient \( \phi(n) \).

Similar quantitative mod \( p \) non-vanishing of Dirichlet \( L \)-values are studied in Burungale-Sun [9]: Let \( \lambda \) be a Dirichlet character of modulus \( N \) and \( (p,NM) = 1 \) and \( (N,M) = 1 \). Removing the condition \( p \nmid \phi(M) \), their result can be formulated as follows.

\[
\# \left\{ \chi \in \hat{\left( \mathbb{Z}/M\mathbb{Z} \right)} \times \mid L(0,\lambda \chi) \not\equiv 0 \pmod{p^{1+v_p(\phi(M))}} \right\} \gg M^{1/2-\epsilon}.
\]

Let us remark that even though Theorem A is not strong enough as the result of Burungale-Sun, it is the first result of this type for modular \( L \)-values with Dirichlet twists as far as we know. We thus believe that the result can be taken as one of evidences that our approach opens up new direction of the research. In fact, the non-vanishing result is a consequence of our main results on the Mazur-Rubin conjecture which will be discussed in the next subsection, which are other examples of prospect for our approach.

### 1.2. Conjectures of Mazur-Rubin

In order to understand the growth of Mordell-Weil ranks of an elliptic curve over \( \mathbb{Q} \) in large abelian extensions, Mazur and Rubin [25] have established several conjectures on statistics for modular symbols based on the numerical computations.

For a random variable \( a \) on a probability space \( X \), we denote the probability, mean, and variance of \( a \) on \( X \) by the notations \( P[a|X] \), \( E[a|X] \), and \( V[a|X] \), respectively. Let us consider the probability spaces with uniform probability:

\[
\Sigma_M = \left\{ \frac{a}{M} \mid 1 \leq a < M, \ (a,M) = 1 \right\}, \ \Omega_M = \bigcup_{n \leq M} \Sigma_M.
\]

Let us regard \( m_f^\pm \) as a random variable on the probability space \( \Sigma_M \) or \( \Omega_M \). Let us set \( m_E^\pm = m_{f_E}^\pm \) for the newform \( f_E \) corresponding to an elliptic curve \( E \) over \( \mathbb{Q} \). The periods \( \Omega_{f_E}^\pm \) can be chosen as the Néron period \( \Omega_E^\pm \). We obtain the \( p \)-integrality of \( m_E^\pm \) when the residual Galois representation \( \overline{\rho}_{E,p} \) of \( E \) is irreducible and \( E \) has good and ordinary reduction at \( p \). Then, Mazur-Rubin [24] have proposed:

**Conjecture B** (Mazur-Rubin). Let \( E \) be an elliptic curve over \( \mathbb{Q} \) of conductor \( N \). Then:

1. The random variable \( m_E^\pm \) on \( \Sigma_M \) follows the asymptotic Gaussian distribution as \( M \) goes to infinity.
(2) For a divisor \( d \) of \( N \), there exist two constants \( C_\ell^\pm \) and \( D_\ell^\pm \), called the variance slope and the variance shift, respectively such that
\[
\lim_{M \to \infty} \mathbb{V}[m_E^\pm | \Sigma_M] - C_\ell^\pm \log M = D_\ell^\pm.
\]

(3) Assume that \( \mathcal{P}_{E,p} \) is irreducible and \( E \) has good and ordinary reduction at \( p \). Then, for any integer \( a \) modulo \( p \)
\[
\lim_{M \to \infty} \mathbb{P}[m_E^\pm \equiv a \pmod{p} | \Sigma_M] = \frac{1}{p}.
\]

Petridis-Risager [30] obtain average or \( \Omega_M \)-versions of the statement (1) for general cuspforms \( f \) of cofinite Fuchsian groups and the statement (2) for congruence subgroup \( \Gamma_0(N) \) with square-free \( N \). They could even give an explicit formula for the constant \( C_\ell^\pm \) as well as \( D_\ell^\pm \) in terms of special values of symmetric square \( L \)-function of \( f \). They further established an interval version of (1), that is, for any interval \( J \subseteq [0,1] \), the variable \( m_f^\pm \) on \( \Omega_M \cap J \) follow the Gaussian distribution asymptotically. Their approach is based on the sophisticated theory of non-holomorphic Eisenstein series twisted by the moments of modular symbols. Their work has been generalised to arbitrary weights by Nordentoft [28]. Based on a different approach, the average version of the statement (1) for arbitrary weights \( k \geq 12 \) and level 1 is obtained by Bettin-Drappeau [6] (See Remark 1.1).

In this article, we present another proof of the average version of Conjecture B for a newform of weight 2 for \( \Gamma_0(N) \) and an arbitrary \( N \) including the statement (3) for an elliptic newform. That is, we show that the statements (1)-(3) hold on \( \Omega_M \) and further obtain some partial results regarding the interval version.

For \( r \in \mathbb{Q} \cap [0,1] \), let \( r = [0; m_1, m_2, \ldots, m_\ell] \) be the continued fraction expansion of \( r \). Let \( \frac{P_i}{Q_i} := [0; m_1, \ldots, m_i] \) be the \( i \)-th convergent of \( r \) with \( P_0 = 0 \) and \( Q_0 = 1 \).

We define an element \( g_i(r) := \left\lfloor \frac{P_i - P_{i-1}}{Q_i - Q_{i-1}} \right\rfloor \in \text{GL}_2(\mathbb{Z}) \). We set \( g(r) := g_\ell(r) \). Note that there is a bijection on \( \Sigma_M \) given by
\[
r = \frac{a}{M} \mapsto r^\ast = \frac{\pi}{M}
\]
where \( \pi \) denotes the inverse of a modulo \( M \). In this article, we use a symbol \( \varphi \) for the map \( \varphi : \Gamma_0(N) \backslash \text{GL}_2(\mathbb{Z}) \to \{0,1\} \). Abusing the notation, let \( \varphi \) be a function on \( \mathbb{Q} \cap [0,1] \) which is given by \( \varphi(r) := \varphi(g(r)) \). Let us set \( \varphi^\ast(r) = \varphi(r^\ast) \). Let us denote by \( \Omega_{M,\varphi} \) the uniform probability space defined by
\[
\Omega_{M,\varphi} := \left\{ \frac{a}{c} \bigg| 1 \leq a < c \leq M, (a,c) = 1, \varphi \left( \frac{a}{c} \right) \neq 0 \right\}.
\]

Regarding \( m_f^\pm \) as a random variable on \( \Omega_{M,\varphi} \), we have the following.

**Theorem C** (Theorem 3.6 and 3.7). Let \( f \) be a newform for \( \Gamma_0(N) \) and of weight 2. Let \( \varphi = \varphi^\ast \). Then,

1. The random variable \( m_f^\pm \) on \( \Omega_{M,\varphi} \) follows the asymptotic Gaussian distribution as \( M \) goes to infinity.

2. There exist constants \( C_f^\pm \) and \( D_f^\pm,\varphi \) such that the variance satisfies
\[
\mathbb{V}[m_f^\pm | \Omega_{M,\varphi}] = C_f^\pm \log M + D_f^\pm,\varphi + O(M^{-\gamma})
\]
for some \( \gamma > 0 \).
(3) Let $f$ be the elliptic newform $f_E$. Assume that $\tilde{\rho}_{E, p}$ is irreducible and $E$ has good and ordinary reduction at $p$. Then, for any integer $a$ modulo $p$ and $e \geq 1$

$$P[m_E^\pm \equiv a \pmod{p^e}] = \frac{1}{p^e} + O(M^{-\delta})$$

for some $\delta > 0$.

Theorem C directly implies Conjecture B in an average sense with specific choices of $\varphi$. For the first and third statement we choose $\varphi \equiv 1$. For a divisor $d$ of $N$, define $\varphi_d \equiv 1$. For a divisor $d$ of $N$, let $\varphi_d(r) = 1$ when $(\delta, N) = d$ and 0 otherwise. Note that $\varphi_d$ is well-defined on $\Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z})$. Since the denominators of $r$ and $r^\ast$ are same, we have $\varphi_d^\ast = \varphi_d$.

Then, the particular choice of the map $\varphi = \varphi_d$ implies that the second statement of the average version of Conjecture B is a special case of our result.

We also obtain an interval version of Theorem C in limited cases: the values of $m_E^\pm(r)$ are asymptotically both normally distributed and equidistributed modulo a prime power, as $r$ varies over $\Omega_{M, \varphi} \cap J$ for an interval $J \subseteq [0, 1]$ and $\varphi = \varphi_N$. We refer to Corollary H below.

1.3. From modular symbols to continued fractions: Manin’s trick. Let us describe how the statistics of continued fractions enters into our discussion on the distribution of modular symbols. Manin [21] noticed that the period integral can be written as

$$\int_0^r f(z)dz = \sum_{i=1}^{\ell} \int_{\frac{Pi}{Q_i} - 1 \cdot \frac{Pi}{Q_i} - 1} f(z)dz = -\sum_{i=1}^{\ell} \int_{g_i(r) \cdot \infty}^{g_i(r) \cdot 0} f(z)dz.$$

For $u \in \Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z})$, let us define

$$c_u(r) := \# \{1 \leq i \leq \ell : g_i(r) \in u \}.$$

We then observe that the modular symbols for a newform $f$ can be expressed by

$$(1.1) \hspace{1cm} m_f^\pm(r) = (1 \pm 1) L_f(\chi) + \sum_{u \in \Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z})} c_u(r)w_u^\pm$$

where

$$w_u^\pm = \frac{1}{\Omega_f^\pm} \left\{ \int_{u \cdot 0}^{\infty} f(z)dz \mp \int_{u \cdot 0}^{\infty} f(z^\prime)dz^\prime \right\} \in \mathbb{Q}_f$$

and $z^\prime = -z$. This identity says that the distribution of modular symbols can be determined by the behavior of a vector-valued random variable

$$\mathbf{c} := (c_u),$$

called a modular partition vector. In fact, the main issue in this article is to study the statistical behavior of $\mathbf{c}$ and to obtain applications to the distribution of modular symbols. Theorem C is a specialisation of results on the distribution of the random variable $\mathbf{c}$ on $\Omega_M$. The probabilistic properties of the variable $\mathbf{c}$ essentially follows from the explicit information on its moment generating function, which will be described below in Theorem F and Theorem G.
1.4. Dynamics of continued fractions: work of Baladi-Vallée. We now describe our approach. It is deeply motivated by the work of Baladi-Vallée [2] on dynamics of continued fraction. Let us briefly outline their result and strategy for the proof.

Regarding the length $\ell$ of continued fraction expansion of a rational number as a random variable on $\Sigma_M$, it has been expected that $\ell$ follows the asymptotic Gaussian distribution as $M$ goes to infinity. The first prominent result goes back to Hensley [17] who obtains a partial result on the problem in an average setting, i.e., one for $\Omega_M$. Baladi-Vallée have proved the average version in full generality, based on the dynamical analysis of Euclidean algorithm.

Baladi-Vallée established the quasi-power behavior of moment generating function $\mathbb{E}[\exp(w\ell) \mid \Omega_M]$, which ensures the asymptotic Gaussian distribution of $\ell$ (Theorem 3.5). More precisely, they study a Dirichlet series whose coefficients are essentially the moment generating functions $\mathbb{E}[\exp(w\ell) \mid \Sigma_n]$:

$$L(s, w) = \sum_{n \geq 1} \frac{c_n(w)}{n^s}, \quad c_n(w) = \sum_{r \in \Sigma_n} \exp(w\ell(r))$$

for two complex variables $s, w$ with $\Re s > 1$ and $|w|$ being sufficiently small. The desired estimate then follows from the Tauberian argument on $L(s, w)$. To this end, they settled the analytic properties of the poles of the Dirichlet series $L(s, w)$ and the estimates on its growth in a vertical strip. A crucial observation made in Baladi-Vallée [2] is that a weighted transfer operator plays a central role in settling the necessary properties of $L(s, w)$.

Let $T : [0, 1] \to [0, 1]$ denote the Gauss map which is given by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

for $x \neq 0$ and $T(0) = 0$. Then, the $i$-th digit of the continued fraction expansion of a real number $x$ is given by $\left\lfloor \frac{1}{T^{i-1}(x)} \right\rfloor$, that is, the digits can be viewed as a trajectory of one dimensional Gauss dynamical system $([0, 1], T)$. They considered the weighted transfer operator defined by

$$H_{s, w} f(x) := \sum_{y \in \Sigma_n} \exp(w|y'(x)|^s) \cdot f(y)$$

for two complex variables $s$ and $w$. A key relation they established is that

$$L(2s, w) = F_{s, w} (I - H_{s, w})^{-1}(0)$$

(1.2)

where $F_{s, w}$ denotes the final operator restricted to the indices $y(x) = \frac{1}{m+1}$ with $n \geq 2$. The crucial properties of Dirichlet series for the Tauberian argument thus directly follow from the spectral properties of transfer operator. In particular, the estimate on the growth of $L(s, w)$ in a vertical strip comes from the Dolgopyat-Baladi-Vallée estimate on the operator norms of $H_{s, w}^n, n \geq 1$.

We follow their framework by finding a certain dynamical system and corresponding transfer operator which naturally describe the analytic properties of Dirichlet series associated to the moment generating functions of the modular partition vectors.
1.5. Dynamics of modular partition vectors. Let us describe the dynamics and the transfer operator for modular partition vectors and state the main results on its spectral properties and the extension of the Dolgopyat estimates.

Let $\Gamma$ be any subgroup in $GL_2(\mathbb{Z})$ of finite index. Let $\Psi \in L^\infty(I_\Gamma)$ to be fixed. Abusing the notation, by the same symbol $\Psi$ let us denote the function $\Psi : r \mapsto \Psi(r^*, g(r))$.

For a random variable $a$ on $\Omega_M$, we define $a^*(r) := a(r^*)$ and call $a^*$ the dual random variable of $a$. Let us consider a Dirichlet series associated to the modular partition vector

$$L_\Psi(s, w) = \sum_{n \geq 1} d_n(w) \frac{d_n(w)}{n^s}, \quad d_n(w) = \sum_{r \in \Sigma_n} \Psi(r) \exp(w \cdot c(r))$$

for $s \in \mathbb{C}$ and $w \in \mathbb{C}^{[GL_2(\mathbb{Z}) : \Gamma]}$. Note that we also have

$$d_n(w) = \sum_{r \in \Sigma_n} \Psi^*(r) \exp(w \cdot c^*(r)).$$

We define a map $\mathbf{T}$ on $I_\Gamma = [0, 1] \times \Gamma \backslash GL_2(\mathbb{Z})$ by

$$\mathbf{T}(x, u) := \left( T(x), u \begin{bmatrix} -m_1(x) & 1 \\ 1 & 0 \end{bmatrix} \right)$$

where $m_1(x)$ denotes the first digit of continued fraction expansion of $x$. We call $\mathbf{T}$ the skewed Gauss map. We would like to emphasize that Manin-Marcolli [22] also considered this skew-product Gauss map to study limiting behavior of modular symbols. For two parameters $s \in \mathbb{C}$ and $w \in \mathbb{C}^{[GL_2(\mathbb{Z}) : \Gamma]}$, we consider the weighted transfer operator associated to the dynamical system $(I_\Gamma, \mathbf{T})$ defined by

$$\mathcal{L}_{s, w} \Psi(x, u) := \sum_{(y, v) \in \mathbf{T}^{-1}(x, u)} \exp[w \cdot \mathbf{I}(v)] \left| \frac{dy}{dx} \right|^s \Psi(y, v)$$

where $\mathbf{I} = (\mathbb{I}_u)_u$ and $\mathbb{I}_u$ for $u \in \Gamma \backslash GL_2(\mathbb{Z})$ denotes a membership function given by $\mathbb{I}_u(g) = 1$ if $g \in u$ and 0 otherwise. Our crucial observation is that the Dirichlet series admits an alternative expression in terms of the weighted transfer operator $\mathcal{L}_{s, w}$:

**Theorem D** (Theorem 4.4). Define $\mathcal{J} \Psi(x, v) = \Psi(1-x, v)$. Then we have

$$L_\Psi(2s, w) = F_{s, w}(\mathcal{I} - \mathcal{L}^2_{s, w})^{-1} \Psi(0, I_2) + F_{s, w} \mathcal{L}_{s, w}(\mathcal{I} - \mathcal{L}^2_{s, w})^{-1} \mathcal{J} \Psi(0, I_2)$$

where $F_{s, w}$ is the final operator restricted to the indices $(y, v) = (\frac{1}{m+2}, u[0 \ 1 \ m])$ with $m \geq 2$.

Observe that under the assumption $\mathcal{J} \Psi = \Psi$ or $\Psi = \Psi^*$, the relation becomes

$$L_\Psi(2s, w) = F_{s, w}(\mathcal{I} - \mathcal{L}^2_{s, w})^{-1} \Psi(0, I_2)$$

which can be viewed as a generalisation of the expression (1.2).

Let $C^1(I_\Gamma)$ be the Banach space of the continuously differentiable functions endowed with the norm

$$||\Psi||_{(t)} := ||\Psi||_0 + \frac{1}{|t|} ||\partial \Psi||_0 \quad (t \neq 0)$$

for $\Psi \in C^1(I_\Gamma)$, where $||.||_0$ denotes the supremum norm and $\partial$ denotes a certain derivative (see §5.1 for a precise definition). The operator $\mathcal{L}_{s, w}$ then acts boundedly.
on $C^1(I_T)$. In view of Theorem D, the poles of Dirichlet series are in bijection with eigenvalues $\lambda_{s,w}$ of transfer operator in a vertical strip. Hence the necessary properties (Proposition 2.2) of $L_\Psi(2s,w)$ for applying the Tauberian theorem follow from the following spectral properties of $L_{s,w}$.

**Theorem E** (Proposition 5.5, 5.8, 5.11 and Theorem 6.5). For $s = \sigma + it$ and $w = x + iy$ for real $\sigma, t, x, y$ with $(\sigma, x)$ in a compact neighborhood of $(1, 0)$,

1. For $\Gamma = \Gamma_0(N)$, the operator $L_{s,w}$ has the dominant eigenvalue $\lambda_{s,w}$ of maximal modulus, which is unique and algebraically simple, and $\lambda_{1,0} = 1$.
2. For a fixed $v > 0$ and $y = (y_u)_u$ with $v < \max_u |y_u| \leq \pi$, the eigenvalues of $L_{s,w}$ are away from 1.
3. (Dolgopyat-Baladi-Vallée) For any $0 < \xi < \frac{1}{2}$, there exists $0 < \eta < 1$ such that
   $$
   ||L_{s,w}^n||_{||t||} \ll \eta^n \cdot \lambda_{s,w}^n ||t||^{\xi}
   $$
   for sufficiently large $||t||$ and all $n \geq 1$.

In order to deduce the uniqueness and simplicity of the dominant eigenvalue $\lambda_{s,w}$, a particular choice of $\Gamma = \Gamma_0(N)$ is required. We refer to Remark 2.3 and Proposition 5.10 for more details. The estimate of the type (3) was first established by Dolgopyat [11] for certain Anosov flows. Baladi-Vallée [2] then extended the result to the Gauss dynamical system. It essentially follows from their modification of Uniform Non-Integrality property for the Gauss map. We refer to §6 for further details.

Our work extends Baladi-Vallée [2] in various ways. Taking $\Gamma$ as the full group $\text{GL}_2(\mathbb{Z})$, their work is a specialisation of ours as the case of level 1. Moreover, we consider more general probabilistic space dependent on $\Psi$ as described in the next subsection. Choosing appropriate $\Psi$ enables us to study the distribution subject to various conditions. Indeed, it enables us to justify the existence of variance slope and shift in Theorem C.(2); and the interval versions as in Corollary H.

**Remark 1.1.** Bettin-Drappeau [6] have generalized the work of Baladi-Vallée in a different direction. They study a wider class of cost functions on the digits of the continued fraction expansions and obtain distribution results on the several crucial examples of quantum modular forms. One of the results is the asymptotic Gaussian distribution of modular symbols for level 1 and arbitrary weights.

### 1.6. Distribution of modular partition vectors.

Let us now state our main results on the distribution of the modular partition vectors.

For a non-negative function $\theta$ on $\Omega_M$ with $\sum_{r \in \Omega_M} \theta(r) \neq 0$, let $\Omega_M,\theta$ be the probability space $\Omega_M$ with the probability density $(\sum_{r \in \Omega_M} \theta(r))^{-1}\theta$. For a non-trivial function $\Psi \in C^1(I_T)$, it can be shown that $\sum_{r \in \Omega_M} \Psi(r)$ is not zero when $M$ is sufficiently large (see Remark 2.8). Hence it makes sense to define $\Omega_{M,\Psi}$ if $\Psi$ is non-negative. For example, we have $\Omega_M,\varphi = \Omega_{M,\varphi}$. Let $C^1(I_T)^+$ be the cone of all non-trivial and non-negative functions in $C^1(I_T)$.

Perron’s formula (Theorem 2.1) for $L_\Psi(s,w)$ together with (1) and (3) of Theorem E, enables us to obtain the quasi-power expression for moment generating function of the random vector $\zeta$ on $\Omega_M,\Psi$, when $w$ is near $0$. A probabilistic result due to Heuberger-Kropf (Theorem 2.11) asserts that such quasi-power behavior implies the asymptotic Gaussian distribution of $\zeta$ on $\Omega_M,\Psi$. In summary, we obtain:

**Theorem F** (Theorem 2.9 and 2.12). Let $\Psi \in C^1(I_{\Gamma_0(N)})^+$. Then:
Theorem G (Theorem 2.14 and Corollary 2.16). Let $\Psi \in C^1(I_{\Gamma_0(N)})^+$ and $x$ near $0$. Then:

1. For a fixed $v > 0$ and $y = (y_n)_n$ with $v < \max_u |y_u| \leq \pi$, we have
   \[ E[\exp(w \cdot \psi) \mid \Omega_{M, \Psi}] \ll M^{-\delta} \cdot \|\Psi\|_{(M^*)} \]
   for some $\delta, \kappa > 0$.

2. For any integer $q > 1$ and $a \in (\mathbb{Z}/q\mathbb{Z})^{[GL_2(\mathbb{Z}) : \Gamma]}$, we have
   \[ P[\psi \equiv a \, (\text{mod } q) \mid \Omega_{M, \Psi}] = q^{-[GL_2(\mathbb{Z}) : \Gamma]} + O(M^{-\delta}) \]
   and also the same formula for $\psi^*$ on $\Omega_{M, \Psi^*}$.

The specialisation $\Psi = 1 \otimes \varphi$ and $w = (zw_u^{1/2})_u$ from (1.1), we are able to deduce the statements (1) and (2) of Theorem C with the help of another probabilistic result due to Hwang (Theorem 3.5).

Hwang’s formula for $L(\psi, s, w)$ together with (2) and (3) of Theorem E, gives us an estimate for $E[\exp(w \cdot \psi) \mid \Omega_{M, \Psi}]$ when $y$ is away from $0$. The equidistribution results follow from an observation that mod $q$ equidistribution statement is usually based on the orthogonality relation between the additive characters $a \mapsto \exp(2\pi i \frac{aq}{q})$, $0 \leq k < q$. In summary, we have:

**Theorem G** (Theorem 2.14 and Corollary 2.16). Let $\Psi \in C^1(I_{\Gamma_0(N)})^+$ and $x$ near $0$.

- For a fixed $v > 0$ and $y = (y_n)_n$ with $v < \max_u |y_u| \leq \pi$, we have
  \[ E[\exp(w \cdot \psi) \mid \Omega_{M, \Psi}] \ll M^{-\delta} \cdot \|\Psi\|_{(M^*)} \]
  for some $\delta, \kappa > 0$.

- For any integer $q > 1$ and $a \in (\mathbb{Z}/q\mathbb{Z})^{[GL_2(\mathbb{Z}) : \Gamma]}$, we have
  \[ P[\psi \equiv a \, (\text{mod } q) \mid \Omega_{M, \Psi}] = q^{-[GL_2(\mathbb{Z}) : \Gamma]} + O(M^{-\delta}) \]
  and also the same formula for $\psi^*$ on $\Omega_{M, \Psi^*}$.

The specialisation $\Psi = 1 \otimes \varphi$ and $w = (z, z, \ldots, z)$ with $Rz$ near $0$ and $3z$ away from $0$, gives us the mod $q$ equidistribution of the length $\ell$ of the continued fractions on $\Omega_{M, \Psi}$. The statement (3) of Theorem C also follows from Theorem G with the corresponding normalised specialisation of $w$.

We conclude with other consequences of Theorem F and G. A useful observation is that the probability space $\Omega_{M, \varphi}$ can be alternatively understood as $\Omega_{M, J} \otimes \varphi$, where $\mathbb{I}_J$ denotes the characteristic function of $J$ for an interval $J \subseteq [0, 1]$. We write $\Omega_{M, \varphi}$ as $\Omega_{M, \varphi, J}$ for short. Since the function $\mathbb{I}_J \otimes \varphi$ does not belong to $C^1(I_{\Gamma_0})$, we make use of a smooth approximation to $\mathbb{I}_J$ in order to obtain the estimate of $E[\exp(w \cdot \psi^*) \mid \Omega_{M, \varphi, J}]$. We obtain the interval version of the previous theorems:

**Corollary H** (Corollary 2.13 and Theorem 2.16). Let $J$ be an interval in $[0, 1]$ and $\varphi$ a function on $\Gamma_0(N) \setminus GL_2(\mathbb{Z})$. Then:

1. The distribution of $\psi^*$ on $\Omega_{M, J, \varphi}$ is asymptotically Gaussian as $M$ goes to infinity.

2. Let $q > 1$ be an integer and $a \in (\mathbb{Z}/q\mathbb{Z})^{[GL_2(\mathbb{Z}) : \Gamma_0(N)]}$. We have
   \[ P[\psi^* \equiv a \, (\text{mod } q) \mid \Omega_{M, J, \varphi}] = \frac{1}{q} + O(M^{-\delta}) \]
   for the same $\delta$ in Theorem G.
Immediate consequences of Corollary H are the interval equidistributions mod $p$ of $m_E^\pm$ and $\ell^\pm$ on $\Omega_{M,J,\varphi}$. Since $J^\ast$ is no longer an interval, we are not able to conclude the same statement for $c$. However, we use the Atkin-Lehner relation (see §3.3) of the modular symbols to convert the equidistribution statement for $m_E^\pm$ to one for $m_E^\ast$, at least on $\Omega_{M,J,\varphi_N}$ (see Remark 3.4).

**Theorem I** (Theorem 3.6 and 3.7). For an interval $J \subseteq [0,1]$:

1. The distribution of $m_E^\pm$ on $\Omega_{M,J,\varphi_N}$ is asymptotically Gaussian as $M$ goes to infinity.

2. For an integer $a$, under the conditions of Theorem C.(3), a prime $p$, $e \geq 1$, there exists a $\delta > 0$ such that

$$\mathbb{P}[m_E^\pm \equiv a (\text{mod } p^e)|\Omega_{M,J,\varphi_N}] = \frac{1}{p^e} + O(M^{-\delta}).$$

Let us remark that an attempt to obtain a result on the interval version of Theorem C.(3) for $\varphi_d$ with $d \neq N$, seems unsatisfactory mainly due to the fact that the Atkin-Lehner conversion does not preserve the space $\Omega_{M,J,\varphi_d}$. In a subsequent paper, we would like to resolve this problem.

Mazur-Rubin, in a private communication, raised a question whether the Gaussian or archimedean, and residual distributions of the modular symbols are correlated. For example, they asked if the moment generating function for the space of the rationals of which modular symbols is congruent to a fixed integer modulo prime, is same as that for the original space without the congruence condition. We answer the latter question in §2.4 and Theorem 3.7.

The paper consists of two major parts: distribution of modular partitions (§2 and §3) and dynamics of the corresponding transfer operator (§5 and §6). In the first half, we prove the limit Gaussian distribution, residual equidistribution of the modular partition vector, and an average version of conjectures of Mazur-Rubin, under the assumption of crucial behaviors of the relevant Dirichlet series in vertical strips. In §4, we reduce the distribution problem to the dynamical ones by introducing the transfer operator associated to the skewed Gauss dynamical system and reconstructing the Dirichlet series in terms of the quasi-inverse of the operator. In the second half of the article, we study spectral properties of the transfer operator and finally obtain the Dolgopyat-Baladi-Vallée estimate in a vertical strip that implies the proofs of the assumptions for the behavior of the Dirichlet series.

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2. Modular partitions of continued fractions

In this section, we prove the limit joint Gaussian distribution and the residual equidistribution of modular partition vector. A key point is that we directly obtain an explicit estimate for the moment generating functions.

Let $\Gamma$ be a subgroup in $\text{GL}_2(\mathbb{Z})$ of finite index. Recall that for $I_{r} = [0,1] \times \Gamma \backslash \text{GL}_2(\mathbb{Z})$, $r \in [0,1] \cap \mathbb{Q}$, and $u \in \Gamma \backslash \text{GL}_2(\mathbb{Z})$, we set $c_u(r) = \# \{1 \leq i \leq \ell : g_i(r) \in u\}$ and $c = (c_u)_u$. Our primary goal is to study the probabilistic behavior of $c(r)$ and $c^\ast(r)$ as $r$ varies over $\Omega_{M,\varphi}$ and $\Omega_{M,\varphi^\ast}$, respectively, for $\Psi \in C^1(I_{r})$. 

2.1. Dirichlet series associated to modular partitions. Recall that a Dirichlet series associated to the modular partition vector and $\Psi \in L^\infty(\Gamma)$ is given by

$$L_\Psi(s, w) = \sum_{n \geq 1} \frac{d_n(w)}{n^s}, \quad d_n(w) = \sum_{r \in \mathcal{B}_n} \Psi(r) \exp(w \cdot c(r))$$

for $s \in \mathbb{C}$ and $w \in \mathbb{C}^{[\text{GL}_2(\mathbb{Z}), \Gamma]}$. Observe that if $d_n(0)$ is non-zero, then

$$\frac{\sum_{n \leq M} d_n(w)}{\sum_{n \leq M} d_n(0)} = \mathbb{E}[\exp(w \cdot \mathfrak{c})|\Omega, \Psi].$$

The average of the coefficients $d_n(w)$ can be studied by the Dirichlet series $L_\Psi(s, w)$ using the following truncated Perron’s formula.

**Theorem 2.1** (Truncated Perron’s Formula, Titchmarsh [38]). Let $F(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ for $Rs := \sigma > \sigma_a$, the abscissa of absolute convergence of $F(s)$. Then for all $D > \sigma_a$, one has

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{D-iT}^{D+iT} F(s) \frac{x^s}{s} ds + O\left(\frac{x^D|F|(D)}{T} + O\left(\frac{A(2x)x \log x}{T}\right)\right) + O\left(A(N)\min\left\{\frac{x}{T|x-N|}, 1\right\}\right)$$

where

$$|F|(\sigma) = \sum_{n \geq 1} \frac{|a_n|}{n^\sigma}$$

for $\sigma > \sigma_a$, $N$ is the nearest integer to $x$, and $a_n = O(A(n))$ with $A(n)$ being non-decreasing.

For tuples $x$ and $y \in \mathbb{R}^{[\text{GL}_2(\mathbb{Z}), \Gamma]}$, let us write

$$w := x + iy$$

and denote $|x| := \max_j |x_j|$. We deform the contour of integration in Perron’s formula in order to detect the pole of $L_\Psi(s, w)$ and to obtain better estimates on $L_\Psi(s, iy)$. When $\Psi \in C^1(\Gamma)$, we can find a vertical strip containing $s = 1$, on which $L_\Psi(2s, w)$ admits a unique pole for $w$ near $0$ and $L_\Psi(s, iy)$ is analytic for $y$ away from $0$. In both cases, the Dirichlet series satisfy the uniform polynomial estimates in the vertical strip. These are our main technical results in this article as elaborated below.

**Proposition 2.2.** For any $0 < \xi < \frac{1}{4}$, we can find $0 < \alpha_0 < \frac{1}{4}$ with the following properties: For any $\hat{\alpha}_0$ with $0 < \hat{\alpha}_0 < \alpha_0$, there exists a neighborhood $W$ of $0$ and a unique analytic function $s_0 : W \to \mathbb{C}$ such that for any $\Psi \in C^1(\Gamma)$ and for all $w \in W$,

1. The value $s_0(0) = 1$ and the Hessian of $s_0$ is non-singular at $w = 0$, and
2. The real part $\Re s_0(w) > 1 - (\alpha_0 - \hat{\alpha}_0)$.
3. Let $I = \Gamma_0(N)$. Then, $L_\Psi(2s, w)$ has only a simple pole at $s = s_0(w)$ in the strip $|\Re s - 1| \leq \alpha_0$ and its residue $E_\Psi(w)$ is analytic on $W$ satisfying

$$|E_\Psi_1(w) - E_\Psi_2(w)| \ll ||\Psi_1 - \Psi_2||_1,$$

for $\Psi_1, \Psi_2 \in C^1(\Gamma)$ and

$$E_\Psi(0) = \frac{1}{2\log 2} \int_{\Gamma} \Psi(x, v) dx dv.$$
Lemma 2.5. We have $|L_\Psi(2s, w)| \ll \max(1, |t|^\xi)|\Psi||_{(\ell)}$ in the strip $|\Re s - 1| \leq \alpha_0$ with $t = 3s$.

For any $0 < \xi < \frac{1}{2}$, the real tuple $x$ near 0, and $v < |y|$ with a fixed $\kappa > 0$, there exists $\alpha_1 > 0$ such that

(5) The series $L_\Psi(2s, w)$ is analytic in the strip $|\Re s - 1| \leq \alpha_1$.

(6) We have $|L_\Psi(2s, w)| \ll \max(1, |t|^\xi)|\Psi||_{(\ell)}$ in the strip $|\Re s - 1| \leq \alpha_1$.

Here $||.||_{(\ell)}$ is the operator norm on $C^1(I_\Gamma)$ introduced before (see §6 for a precise definition). We postpone the proof of Proposition 2.2 to the end of present article after discussing dynamical preliminaries. For reader’s convenience, we briefly sketch how Proposition 2.2 can be obtained. In §4–§6, we introduce the skewed Gauss map and the associated transfer operator on $C^1(I_\Gamma)$; and settle an explicit relation between the resolvent of the operator and Dirichlet series associated to the modular partition vector $c$. Thus, the desired analytic properties of $L_\Psi(2s, w)$ for Perron’s formula can be obtained by the dynamical/spectral analysis of the transfer operator via this key relation.

Remark 2.3. In present article, a particular specialisation of $\Gamma$ to $\Gamma_0(N)$ is required only to conclude the simplicity and uniqueness of the pole in Proposition 2.2 (3). We refer to Proposition 5.10.

Remark 2.4. Since all the arguments depend not on the function $\Psi$ but on the transfer operator, the constants $\alpha_0$, $\tilde{\alpha}_0$, $\alpha_1$, $\xi$, and all the implicit constants are all independent of $\Psi$. This independence plays a crucial role in obtaining the interval version of residual equidistribution result in Theorem 2.16.

2.2. Joint Gaussian distribution of modular partitions. In this subsection, we obtain an explicit quasi-power estimate for moment generating function of the modular partition vector and thus show the limit joint Gaussian distribution.

We apply Theorem 2.1 to the Dirichlet series $L_\Psi(2s, w)$. Then, Proposition 2.2 makes it legitimate to deform the contour integration in the Perron formula. Let us first check the condition of Theorem 2.1.

Lemma 2.5. For some $c > 0$, we have

$$d_n(w) \ll n^{1+c|x|}||\Psi||_0.$$  

Proof. Note that $w \cdot c(r) \ll \ell(r)|x|$. Since it is well-known that there exists a constant $c > 0$ such that $\ell(r) \leq c \log n$ for $r \in \Sigma_n$, we have

$$d_n(w) \ll \sum_{r \in \Sigma_n} |\Psi(r)| \exp(\ell(r)|x|) \ll n^{1+c|x|}||\Psi||_0.$$  

This finishes the proof. \qed

Hence we take $A(M)$ in Perron’s formula with $A(M) = M^{1+c|x|}$. The following is one of our main results which leads us to the asymptotic Gaussian behavior of the variable $c$.

Proposition 2.6. There exist constants $0 < \delta < 2$, $\kappa > 0$, and a neighborhood $W$ of 0 such that for all $\Psi \in C^1(I_\Gamma)$, we have

$$\sum_{n \in M} d_n(w) = \frac{E_\Psi(w)}{s_0(w)} M^{2s_0(w)} + O(M^\delta ||\Psi||_{(M^\kappa)}).$$  

(2.1)
The Perron formula in Theorem 2.1, we have $M$ the exponent of $\exp$ independent of $\Psi$. We choose $\tilde{\alpha}_0$ with

$$\frac{8}{59} \alpha_0 < \tilde{\alpha}_0 < \alpha_0$$

and set

$$T = M^{2\alpha_0 + 4\tilde{\alpha}_0}.$$ 

Note that $E_{\Psi}(w)$ is bounded in the neighborhood $W$ since $s_0(0) = 1$. Then, the error terms are bounded as follows.

The error term I is equal to $O(M^{2(1-2\tilde{\alpha}_0)}||\Psi||_{(T)})$ and by Proposition 2.2, the exponent of $M$ satisfies $2(1 - 2\tilde{\alpha}_0) < 2$.

By Lemma 2.5, for any $0 < \varepsilon < \frac{2\alpha_0}{T}$, we can take $W$ from Proposition 2.2 small enough to have $c|\xi| < \varepsilon/2$ so that $A(M) = O(M^{1+\varepsilon/2})$ and $\log M \ll M^{\varepsilon/2}$. Hence, the exponent of $M$ in the error term II is equal to

$$1 + (1 + c|\xi|) + \frac{\varepsilon}{2} - (2\alpha_0 + 4\tilde{\alpha}_0) \leq 2 - \frac{23}{4} \tilde{\alpha}_0 < 2.$$

Similarly the error term III is equal to $O(M^{1+\varepsilon/2})$, so the exponent satisfies

$$1 + \frac{\varepsilon}{2} < 1 + \frac{1}{4} \tilde{\alpha}_0 < 2.$$ 

Also for any $0 < \xi < \frac{1}{5}$, we have $|L_{\Psi}(2\xi, w)| \ll |\Re s|^\xi||\Psi||_{(\Re s)}$ by Proposition 2.2. Hence, the error term IV is $O(M^{2(1-\alpha_0)}T^\xi||\Psi||_{(T)})$ and thus the exponent of $M$ is
equal to
\[ 2(1 - \alpha_0) + (2\alpha_0 + 4\hat{\alpha}_0)\xi < 2 - \frac{4}{5}(2\alpha_0 - \hat{\alpha}_0) < 2. \]

The last term \( V \) is \( O(T^\xi - 1 \cdot M^{2(1+\alpha_0)}\log M||\Psi||_{(T)}) \), hence the exponent of \( M \) satisfies
\[ (2\alpha_0 + 4\hat{\alpha}_0)(\xi - 1) + 2(1 + \alpha_0) + \frac{\epsilon}{2} < 2 - \left( -\frac{2}{5}\alpha_0 + \frac{59}{20}\hat{\alpha}_0 \right) < 2. \]

In total, setting \( \kappa = 2\alpha_0 + 4\hat{\alpha}_0 \) and
\[ \delta = \min \left( 2 - \frac{23}{4}\hat{\alpha}_0, 1 + \frac{1}{4}\hat{\alpha}_0, 2 - \frac{4}{5}(2\alpha_0 - \hat{\alpha}_0), 2 - \left( -\frac{2}{5}\alpha_0 + \frac{59}{20}\hat{\alpha}_0 \right) \right), \]
we conclude the proof. \( \square \)

Observe that Proposition 2.6 implies that
\[ (2.2) \quad \sum_{n \leq M} d_n(0) = E_\Psi(0)M^2 + O(M^\delta||\Psi||_{(M)}) \]

Hence, from Proposition 2.2.(3) we have
\[ (2.3) \quad \lim_{M \to \infty} \frac{1}{|\Omega_M|} \sum_{r \in \Omega_M} \Psi(r, g(r^*)) = \frac{\pi^2}{12\log 2} \int_{I_\Gamma} \Psi(x,v)dx\,dv \]
for all \( \Psi \in C^1(I_\Gamma) \). Here recall that \( |\Omega_M| \sim \frac{\delta}{4\pi} M \). Applying Weyl’s criterion on the equidistribution, we obtain a non-trivial consequence of Proposition 2.6:

\[ \text{Corollary 2.7.} \quad \text{The set } \{(r, g(r^*)) | r \in \Omega_M \} \text{ is equidistributed with respect to the measure } \frac{\pi^2}{12\log 2}dx\,dv \text{ in } I_\Gamma. \text{ In particular, } \Omega_{M,\varphi} \text{ is equidistributed in } [0,1] \text{ for any function } \varphi. \]

\[ \text{Remark 2.8.} \quad \text{The expression (2.3) implies that } \sum_{r \in \Omega_M} \Psi(r) \text{ is not zero if } M \text{ is sufficiently large unless } \Psi \text{ is trivially zero. Hence, we are able to define a density function } (\sum_{r \in \Omega_M} \Psi(r))^{-1} \Psi \text{ on } \Omega_M \text{ for } \Psi \in C^1(I_\Gamma)^+. \]

Now we show that the moment generating function of \( \epsilon \) on \( \Omega_{M,\varphi} \) has the quasi-power behavior for \( \Psi \in C^1(I_\Gamma)^+ \) or \( \Psi = I_J \otimes \varphi \) for an interval \( J \) in \([0,1]\) and a function \( \varphi \) on \( \Gamma_0(N)\GL_2(\mathbb{Z}) \). Since \( I_J \otimes \varphi \) does not belong to \( C^1(I_\Gamma) \), we approximate the moment generating function using a smooth approximation of \( I_J \otimes \varphi \).

\[ \text{Theorem 2.9.} \quad \text{Let } \Psi \in C^1(I_{\Gamma_0(N)})^+ \text{ or } \Psi = I_J \otimes \varphi. \text{ Then, there exist a neighborhood } W \text{ of } 0, \text{ an analytic function } B_\Psi \text{ on } W, \text{ and a constant } 0 < \gamma < \alpha_0 \text{ with } \alpha_0 \text{ from Proposition 2.2, such that } B_\Psi \text{ is non-vanishing on } W \text{ and} \]
\[ \mathbb{E}[\exp(w \cdot \epsilon)|\Omega_{M,\varphi}] = \frac{B_\Psi(w)}{B_\Psi(0)}M^{2(s_0(w) - s_0(0))}(1 + O(M^{-\gamma})) \]

with \( s_0(w) \) from Proposition 2.2 (1) and \( w \in W \). The implicit constant and the constant \( \gamma \) are independent of \( w \in W \) and \( \Psi \).

\[ \text{Proof.} \quad \text{The statements for } \Psi \in C^1(I_\Gamma)^+ \text{ follows directly from Proposition 2.6 with } B_\Psi(w) = E_\Psi(w)/s_0(w). \text{ Moreover we can observe that the neighborhood } W \text{ is independent of } \Psi. \]

Let us consider the case of \( \Psi = I_J \otimes \varphi \). For \( Q > 0 \), let \( h_{Q,J} \) be a smooth approximation of \( I_J \) such that \( ||h_{Q,J}||_0 \leq Q \) and the support of \( I_J - h_{Q,J} \) consists
of intervals and their total length is $\ll Q^{-1}$. Let us set $\Psi_{Q,J,\varphi} = h_{Q,J} \otimes \varphi$. Then $\Psi_{Q,J,\varphi}$ belongs to $C^1(I_1)$. It is easy to see

$$\sum_{r \in \Omega_M} (\Psi(r) - \Psi_{Q,J,\varphi}(r)) \exp(w \cdot c) \ll \#(\Omega_M \cap \text{supp}(I_J - h_{Q,J})) \cdot M^{|x|}$$

One can easily show that the last quantity is $\ll M^{2+|x|}Q^{-1}$. By Proposition 2.6, there exists a neighborhood $W_1$ of $0$ such that for all $w \in W_1$ and all $Q \gg 1$, we have

$$\sum_{r \in \Omega_M} \Psi(r) \exp(w \cdot c(r)) = \frac{E_{\Psi_{Q,J,\varphi}}(w)}{s_0(w)} M^{2s_0(w)} + O(M^{2+|x|}Q^{-1}) + O(M^5 + QM^{-\kappa}).$$

Let us define $E_{\Psi}(w)$ by the expression (6.3). Then, we have

$$|E_{\Psi}(w) - E_{\Psi_{Q,J,\varphi}}(w)| \ll Q^{-1}$$

for all $w \in W_1$. Now we choose a neighborhood $W_2 \subseteq W_1$ so that $|x| < \kappa$ for all $w \in W_2$. In total, we have

$$\sum_{r \in \Omega_M} \Psi(r) \exp(w \cdot c(r)) = \frac{E_{\Psi}(w)}{s_0(w)} M^{2s_0(w)}(1 + O(Q^{-1})) + O(M^{2+\kappa}Q^{-1}) + O(M^5 + QM^{-\kappa}).$$

Setting $Q = M^{1+\kappa}$ and $\gamma = \min(1, 2 - \delta)$, we have

$$\sum_{r \in \Omega_M} \Psi(r) \exp(w \cdot c(r)) = \frac{E_{\Psi}(w)}{s_0(w)} M^{2s_0(w)}(1 + O(M^{-\gamma})).$$

Since $E_{\Psi}(0) \neq 0$, there exists a $W \subseteq W_2$ such that $E_{\Psi}$ is non-vanishing on $W$. This concludes the proof of theorem. \hfill \Box

Let us record a consequence of Theorem 2.9, namely, the behavior of the moment generating function of $c^*$ on $\Omega_{M,J,\varphi}$. Let us recall that $\Psi(r) = \Psi(r^*, g(r))$ and that $\Omega_{M,J,\varphi} = \Omega_{M,\varphi} \cap J$ is equal to $\Omega_{M,J,\varphi} \otimes \varphi$.

**Corollary 2.10.** There exist a neighborhood $W$ of $0$, an analytic function $B_{J,\varphi}$ on $W$, and a constant $\gamma > 0$ such that $B_{J,\varphi}$ is non-vanishing on $W$ and

$$E[\exp(w \cdot c^*)|\Omega_{M,J,\varphi}] = \frac{B_{J,\varphi}(w)}{B_{J,\varphi}(0)} M^{2(s_0(w) - s_0(0))}(1 + O(M^{-\gamma}))$$

with $s_0(w)$ from Proposition 2.2 (1) and $w \in W$. The implicit constant is independent of $w \in W$.

We would like to mention that we have used a different version of Perron’s formula (Theorem 2.1) from one used in Baladi-Vallée [2]. The current version directly leads us to get the desired estimate for the moment generating function without the additional smoothing process of Baladi-Vallée. Please see Lee-Sun [19] for the relevant discussion for the length of continued fractions.

The following probabilistic result ensures that the asymptotic normality of a sequence of random vectors can be followed by the quasi-power behavior of their moment generating function.
Theorem 2.11 (Heuberger-Kropf [15]). Suppose that the moment generating function for a sequence $X_N$ of $m$-dimensional real random vectors on spaces $\Xi_N$ satisfies the quasi-power expression

$$\mathbb{E}[\exp(w \cdot X_N) | \Xi_N] = \exp(\beta_N U(w) + V(w))(1 + O(\kappa_N^{-1}))$$

with $\beta_N, \kappa_N \to \infty$ as $N \to \infty$, and $U(w), V(w)$ analytic for $w = (w_i) \in \mathbb{C}^m$ with $|w|$ being sufficiently small. If the Hessian $H_U(0)$ of $U$ at $0$ is non-singular, then the distribution of $X_N$ is asymptotically normal with speed of convergence $O(\kappa_N^{-1} + \beta_N^{-1/2})$. In other words, for any $x \in \mathbb{R}^m$

$$\mathbb{P}\left[ \frac{X_N - \theta U(0) \beta_N}{\sqrt{\beta_N}} \leq x \left| \Xi_N \right. \right] = \frac{1}{(2\pi)^{m/2} \sqrt{\det H_U(0)}} \int_{t \leq x} \exp \left( -\frac{1}{2} t \cdot H_U(0)^{-1} \cdot t \right) dt + O \left( \frac{1}{\kappa_N} + \frac{1}{\sqrt{\beta_N}} \right)$$

where $t \leq x$ means $t_j \leq x_j$ for all $1 \leq j \leq k$ and the $O$-term being uniform in $x$.

Let $U(w) = 2(s_0(w) - s_0(0))$ and $V(w) = \log \frac{B(w)}{B(0)}$ with $s_0$ and $B$ from Theorem 2.9. By Proposition 2.2, both $U$ and $V$ are independent of $M$, analytic for sufficiently small $w$ and the Hessian $H_U(0) = H_{2s_0}(0)$ is non-singular. Hence from Theorem 2.11 and Theorem 2.9, we obtain the following central limit theorem for the modular partition vectors.

Theorem 2.12. Let $\Psi \in C^1(I_{\kappa(N)})^+$. The distribution of $\ell$ on $\Omega_{M, \Psi}$ is asymptotically normal. The same is true for the dual modular partition vector $\ell^*$ on $\Omega_{M, \Psi^*}$.

From Corollary 2.10, we obtain:

Corollary 2.13. The dual modular partition vector $\ell^*$ on $\Omega_{M, \Psi}$ follow asymptotic Gaussian distribution as $M$ goes to infinity.

2.3. Residual equidistribution of modular partitions. In this subsection, we show that for any integer $q > 1$, mod $q$ values of the modular partition vector are equidistributed. Moreover, we establish an interval version of the result for $\ell^*$, i.e., residual equidistribution of $\ell^*$ on $\Omega_{M, \Psi} \cap \Omega_{J, \Psi}$, where $J$ is any interval in $[0, 1]$.

We have the following estimate for $\mathbb{E}[\exp(y \cdot \ell) | \Omega_{M, \Psi}]$ when $|y|$ is away from $0$. An important part is that the exponent of $M$ and implicit constant in the estimation are independent of $\Psi$ (see Remark 2.4).

Theorem 2.14. For a fixed $v > 0$, $y$ with $v < |y| \leq \pi$, and $x$ near $0$, there exists $\gamma_1 > 0$ such that for $\Psi \in C^1(I_T)^+$ we have

$$\mathbb{E}[\exp(w \cdot \ell) | \Omega_{M, \Psi}] \ll M^{-\gamma_1} \|\Psi\|_{(\Lambda^\kappa_1)} \|\Psi\|_{L_1}^{-1}.$$ 

The implicit constant is independent of $\Psi$.

Proof. By the statements (5) and (6) of Proposition 2.2, $L_q(2s, w)$ is analytic in the rectangle $U_T$ with vertices $1 + \alpha_1 + iT, 1 - \alpha_1 + iT, 1 - \alpha_1 - iT, 1 + \alpha_1 - iT$. Cauchy’s residue theorem yields

$$\frac{1}{2\pi i} \int_{U_T} L_q(2s, w) \frac{M^{2s}}{s} ds = 0$$
and together with Perron’s formula in Theorem 2.1, we have
\[
\sum_{n \leq M} d_n(w) = O \left( \frac{M^{2(1+\alpha_1)}}{T} ||\Psi||_{(T)} + O \left( \frac{(A(2M)M \log M)}{T} + O(A(M)) \right) \right.
\]
\[
+ O \left( \int_{1-\alpha_1 - iT}^{1-\alpha_1 + iT} |L(2s,w)| \frac{M^{2(1-\alpha_1)}}{|s|} ds \right)
\]
\[
+ O \left( \int_{1-\alpha_1 \pm iT}^{1+\alpha_1 \pm iT} |L(2s,w)| \frac{M^{2R_s}}{T} ds \right).
\]

We denote it briefly by \( \sum_{n \leq M} d_n(w) = I + II + III + IV + V \). Taking
\[
T = M^{5\alpha_1},
\]
the error terms are estimated as follows.

The error term I is simply \( O(M^{2-3\alpha_1} ||\Psi||_{(T)}) \). By Lemma 2.5, for any \( 0 < \varepsilon_1 < \frac{\alpha_1}{2} \), we can take \( A(M) = O(M^{1+\varepsilon_1/2} ||\Psi||_0) \) and \( \log M \ll M^{\varepsilon_1/2} \). Then, the exponent of \( M \) in the error term II is equal to
\[ 2 - \varepsilon_1 - 5\alpha_1 < 2 - \frac{19}{2} \alpha_1 \]
and we thus have \( II = O(M^{2-\frac{19}{2} \alpha_1} ||\Psi||_0) \). The error term III is equal to \( O(M^{1+\alpha_1/4} ||\Psi||_0) \).

Proposition 2.2 states that for any \( 0 < \xi < \frac{1}{2} \), we have the uniform polynomial bound \( |L(2s,w)| \ll |t|^{\xi} ||\Psi||_{(t)} \) when \( |y| \) is far from 0. Thus, the error term IV is
\[
O \left( T^{\xi} M^{2(1-\alpha_1)} ||\Psi||_{(T)} \right)
\]
and the exponent of \( M \) is equal to
\[ 2(1 - \alpha_1) + 5\alpha_1 \xi < 2 - \alpha_1. \]
The last term \( V \) is \( O \left( T^{\xi - 1} \cdot M^{2(1+\alpha_1)} \log M ||\Psi||_{(T)} \right) \), hence the exponent of \( M \) satisfies
\[ 5\alpha_1 (\xi - 1) + \frac{\varepsilon_1}{2} + 2 + 2\alpha_1 < 2 - \frac{7}{4} \alpha_1. \]

By taking
\[ \gamma_0 = \max \left( 2 - 3\alpha_1, 2 - \frac{19}{2} \alpha_1, 1 + \frac{\alpha_1}{4} \right) \]
which is strictly less than 2, we have \( \sum_{n \leq M} d_n(w) = O(M^{\gamma_0} ||\Psi||_{(T)}) \). Recall that we have \( \sum_{n \leq M} d_n(w) \gg M^2 ||\Psi||_{L^2} \) from Theorem 2.9. Therefore, we obtain the proof of the statement for some \( 0 < \gamma_1 < 2 - (\gamma_0 + \gamma) \).

An interval version of the residual equidistribution of the dual variable \( c^* \) is as follows:

**Corollary 2.15.** Let \( J \) be an interval in \([0,1)\) and \( \varphi \) a function on \( \Gamma_0(N) \backslash \text{GL}_2(\mathbb{Z}) \). For a fixed \( v > 0, \ y \) with \( v < |y| \leq \pi, \ x \) near \( 0 \), there exists \( \gamma_2 > 0 \) such that we have
\[
\mathbb{E}[\exp(w \cdot c^*)|\Omega_{M,J,\varphi}] \ll M^{-\gamma_2}.
\]
Proof. Let $\Psi_{Q,J,\varphi} = h_{Q,J} \otimes \varphi^*$ be the function in the proof of Proposition 2.9. As before, we obtain
$$\sum_{r \in \Omega_M} (\Psi^*(r) - \Psi_{Q,J,\varphi}(r)) \exp(w \cdot c^*) \ll M^2 Q^{-1}.$$ Taking $0 < v < q^{-1}$ in Theorem 2.14, for $s \neq 0$, again we have
$$\sum_{r \in \Omega_M} \Psi_{Q,J,\varphi}(r) \exp(w \cdot c^*(r)) \ll |\Omega_{M,Q,J,\varphi}|(M^{-\gamma_1} + M^{-\gamma_1-5\alpha_1}Q).$$ From (2.2), we obtain $|\Omega_{M,Q,J,\varphi}| \ll M^2 \ll |\Omega_{M,J}|$ and hence we can conclude that
$$\mathbb{E}[\exp(w \cdot c^*)]|\Omega_{M,Q,J,\varphi}| \ll M^{-\gamma_1} + Q^{-1} + M^{-\gamma_1-5\alpha_1}Q.$$ Setting $Q = M^{(\gamma_1+5\alpha_1)/2}$ and $\gamma_2 = \min(\gamma_1, (\gamma_1 + 5\alpha_1)/2)$, we finish the proof. $\square$

Let us now mention two consequences of Theorem 2.14 and Corollary 2.15, namely the residual equidistributions of $c$ on $\Omega_{M,\Psi}$ and $c^*$ on $\Omega_{M,\varphi} \cap J$.

**Theorem 2.16.** Let $\Psi \in C^1(\Gamma_0(N))^+$. For any integer $q > 1$ and a tuple of integers $a \in (\mathbb{Z}/q\mathbb{Z})^{[\text{GL}_2(\mathbb{Z})] \cap \Gamma_0(N)}$, we have
$$\mathbb{P}[c \equiv a \pmod{q}|\Omega_{M,\Psi}] = q^{-[\text{GL}_2(\mathbb{Z})] \cap \Gamma_0(N)} + O(M^{-\gamma_1})$$ with $\gamma_1 > 0$ from Theorem 2.14. The same is true for $c^*$ on $\Omega_{M,\varphi^*}$. Furthermore, there exists $\gamma_1 \geq \gamma_2 > 0$ such that we obtain
$$\mathbb{P}[c^* \equiv a \pmod{q}|\Omega_{M,\varphi^*}] = q^{-[\text{GL}_2(\mathbb{Z})] \cap \Gamma_0(N)} + O(M^{-\gamma_2}).$$

**Proof.** We write $k = [\text{GL}_2(\mathbb{Z}) : \Gamma_0(N)]$ for brevity. Recall that $c(r), c^*(r) \in \mathbb{Z}^k$ as $r$ varies over $\Omega_M$. Then for $a \in (\mathbb{Z}/q\mathbb{Z})^k$, we have
$$\mathbb{P}[c \equiv a \pmod{q}|\Omega_{M,\Psi}] = \sum_{m \in \mathbb{Z}^k} \mathbb{P}[c = m|\Omega_{M,\Psi}]$$
$$= \sum_{m \in \mathbb{Z}^k} \left( \frac{1}{q^k} \sum_{s \in (\mathbb{Z}/q\mathbb{Z})^k} \exp \left( \frac{2\pi i}{q} s \cdot (m - a) \right) \right) \mathbb{P}[c = m|\Omega_{M,\Psi}]$$
$$= \frac{1}{q^k} \sum_{s \in (\mathbb{Z}/q\mathbb{Z})^k} e^{-\frac{2\pi i}{q} s \cdot a} \mathbb{E} \left[ \exp \left( \frac{2\pi i}{q} y \cdot \gamma \right) |\Omega_{M,\Psi} \right].$$ We then split the summation into two parts: $s = 0$ and $s \neq 0$. The term corresponds to $s = 0$ is the main term which is $q^{-k}$. For the sum over $s \neq 0$, taking $0 < v < q^{-1}$ in Theorem 2.14, we obtain
$$\mathbb{E} \left[ \exp \left( \frac{2\pi i}{q} y \cdot \gamma \right) |\Omega_{M,\Psi} \right] \ll M^{-\gamma_1},$$ which gives the first statement. The second assertion of the theorem follows in the same way using Corollary 2.15. $\square$

Specialising $w$, we obtain:

**Corollary 2.17.** Let $\Psi \in C^1(\Gamma_0(N))^+$. Let $y_0 \in \mathbb{Z}^{[\text{GL}_2(\mathbb{Z})] \cap \Gamma_0(N)}$ such that one of coordinates of $y_0$ is relatively prime to $q$. Then, $y_0 \cdot c$ is equidistributed modulo $q$ on $\Omega_{M,\Psi}$ and so is $y_0 \cdot c^*$ on $\Omega_{M,\varphi^*}$ or $\Omega_{M,\varphi} \cap J$. $\square$
Proof. Let \( a \in \mathbb{Z}/q\mathbb{Z} \) and \( H \) be the collection of solutions \( a \) for \( y_0 \cdot a \equiv a \pmod{q} \). We then have

\[
P[y_0 \cdot c' \equiv a \pmod{q} | \Omega_{M, \Psi}] = \sum_{a \in H} P[c' \equiv a \pmod{q} | \Omega_{M, \Psi}].
\]

Due to the condition on the coordinates of \( y_0 \), we have \( |H| = q^{[\text{GL}_2(\mathbb{Z}) : \Gamma_0(N)]} - 1 \) and we get the desired statement from Theorem 2.16. We apply this argument again, with \( \Omega_{M, \Psi} \) replaced by \( \Omega_{M, J, \psi} \), to conclude the proof.

By taking \( y_0 = (1, 1, \cdots, 1) \), we obtain an immediate consequence of Corollary 2.17: for a function \( \varphi \), we have

\[ P_M[c' \equiv a \pmod{q} | \Omega_{M, J, \psi}] = q^{-1} + o(1). \]

Obviously, this implies the mod \( q \) equidistribution of \( \ell \) on \( \Omega_{M, J, \psi} \). However, this has nothing to do with the interval version as mentioned before. We would like to study the residual equidistribution of \( \ell \) on \( \Omega_{M, J, \psi} \) in a subsequent paper.

2.4. Non-correlation between the archimedean and mod \( p \) distributions.

In this section we present a result related to the question of Mazur-Rubin about whether the Gaussian distribution of \( c \) is correlated to the residual distribution.

For \( \Psi \in C^1(I_{\Gamma_0(N)}^+) \), let \( \Omega_{M, \psi} \cap (c \equiv a(q)) \) be the probability space \( \{ r \in \Omega_M | c(r) \equiv a \pmod{q} \} \) with the probability density \( \left( \sum_{r \in \Omega_M \cap \{ r \equiv a(q) \}} \Psi(r) \right)^{-1} \). We are able to show:

**Theorem 2.18.** Let us set \( k = [\text{GL}_2(\mathbb{Z}) : \Gamma_0(N)] \). For \( \Psi \in C^1(I_{\Gamma_0(N)}^+) \), an integer \( q > 1 \), \( w \) near \( 0 \), and a tuple \( a \), we have

\[
\sum_{r \in \Omega_M \cap \{ r \equiv a(q) \}} \Psi(r) \exp(w \cdot c(r)) = \frac{1}{q^k} \sum_{r \in \Omega_M} \Psi(r) \exp(w \cdot c(r)) (1 + O(M^{-\gamma})).
\]

In particular, we obtain

\[
\mathbb{E}[\exp(w \cdot c) | \Omega_{M, \psi} \cap (c \equiv a(q))] = \mathbb{E}[\exp(w \cdot c) | \Omega_{M, \psi}](1 + O(M^{-\gamma})).
\]

**Proof.** Note that we have

\[
\sum_{r \in \Omega_M \cap \{ r \equiv a(q) \}} \Psi(r) \exp(w \cdot c(r)) = \frac{1}{q^k} \sum_{r \in \Omega_M} \exp\left( \frac{2\pi i}{q} s \cdot (c(r) - a) \right) \Psi(r) \exp(w \cdot c(r))
\]

\[
= \frac{1}{q^k} \sum_{s \in (\mathbb{Z}/q\mathbb{Z})^k} e^{-\frac{2\pi i}{q} s \cdot a} \sum_{r \in \Omega_M} \Psi(r) \exp\left( \frac{2\pi i}{q} s + w \cdot c(r) \right)
\]

Splitting the sum over \( s \) in the last expression into two parts, \( s = 0 \) and \( s \not= 0 (\pmod{q}) \), it is equal to

\[
\frac{1}{q^k} \sum_{r \in \Omega_M} \Psi(r) \exp(w \cdot c(r)) + \frac{1}{q^k} \sum_{s \not= 0(q)} e^{-\frac{2\pi i}{q} s \cdot a} \sum_{r \in \Omega_M} \Psi(r) \exp\left( \frac{2\pi i}{q} s + w \cdot c(r) \right)
\]

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By Theorem 2.14, for $\gamma_1 > 0$ the last expression is equal to
\[
\frac{1}{q^2} \sum_{r \in \Omega_{s,t}} \Psi(r) \exp(w \cdot c(r)) + O(M^{-\gamma_1}).
\]
Hence we finish the proof. \(\Box\)

Let $y_0$ be the vector in Corollary 2.17. By the same argument as the proof of Corollary 2.17, we also have similar type of non-correlation between the two distributions of $y_0 \cdot c$:

**Corollary 2.19.** For a complex $z$ near 0, we have
\[
\mathbb{E}[\exp(z y_0 \cdot c) \Omega_{M,\psi} \cap (y_0 \cdot c \equiv a(q))] = \mathbb{E}[\exp(z y_0 \cdot c) \Omega_{M,\psi}](1 + O(M^{-\gamma_1})).
\]

### 3. Distribution of modular symbols

In this section, we show that the modular symbols are non-degenerate specialisation of the modular partition vectors in both zero and positive characteristics. Using this, we deduce the distribution results on the modular symbols from those on the modular partitions.

#### 3.1. Involution on de Rham cohomology.

Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. For two cusps $r, s$ in $\mathbb{P}^1(\mathbb{Q})$, we write $\{r, s\}_\Gamma$ for the relative homology class corresponding to the projection to $X_{\Gamma}$ of the geodesic on the upper-half plane $\mathcal{H}$ connecting $r$ to $s$. For $\Gamma = \Gamma_1(N)$, let us set $\{r, s\}_N = \{r, s\}_{\Gamma_1(N)}$.

Let us denote by $H^1_{\text{dR}}(X_{\Gamma})$ the first de Rham cohomology of the modular curve $X_{\Gamma}$. Let us assume that $\Gamma$ is normalised by $j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (e.g. $\Gamma = \Gamma_1(N)$). We define an operator $\iota$ on $\gamma \in \Gamma$ and $z \in \mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ by
\[
\gamma' = j \gamma j \in \Gamma \text{ and } z' = -z \in \mathcal{H}^*.
\]

Then, the action of $\iota$ yields a well-defined involution on $X_{\Gamma}$. Let $S_2(\Gamma)$ be the space of cuspforms of weight 2 for $\Gamma$. The involution $\iota$ then has an action on $H^1_{\text{dR}}(X_{\Gamma})$ and hence on $S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$ via the Hodge decomposition of the de Rham cohomology group.

\[
S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \simeq H^1_{\text{dR}}(X_{\Gamma}).
\]

For $h \in S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$, we have $h^\iota(z) = h(z')$. The involution $\iota$ interchanges $S_2(\Gamma)$ and $\overline{S_2(\Gamma)}$. Moreover, the involution $\iota$ is normal with respect to the cap product
\[
\cap : H_1(X_{\Gamma}, \mathbb{Z}) \times H^1_{\text{dR}}(X_{\Gamma}) \to \mathbb{C}, \ (\xi, \omega) \mapsto \xi \cap \omega = \int_{\xi} \omega.
\]

The cap product can be interpreted as follows. For $f \in S_2(\Gamma)$, $g \in \overline{S_2(\Gamma)}$, and $\{r, s\}_\Gamma \in H_1(X_{\Gamma}, \mathbb{Q})$, set
\[
\langle \{r, s\}_\Gamma, (f, g) \rangle = \int_r^s f(z)dz + \int_r^s g(z)dz'.
\]

Then it is known that the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate. (see Merel [26]). Note that the modular symbol $m_f^\iota(r)$ can be understood as the above pairing (3.2) between a relative homology class $\{r, i\infty\}_N$ and a de Rham cohomology class $(f, \mp f')$. 
Let $f$ be a newform for $\Gamma_0(N)$ of weight 2, i.e., cuspform for $\Gamma_1(N)$ with the trivial Nebentypus. Let us recall the expression (1.1) that the modular symbols are expressed as

$$m_f^\pm(r) = \frac{(1 \pm 1)L(1,f)}{\Omega_f^\pm} + w_f^\pm \cdot c(r).$$

where $w_f^\pm := (w_u^\pm) \in C^{[GL_2(\mathbb{Z})]:\Gamma}$ and

$$w_u^\pm := \frac{1}{\Omega_f} \int_{u,0}^{w,\infty} (f(z)dz) \in \mathbb{Q}_f.$$  

Then, we have

**Proposition 3.1.** For any $f \in S_2(\Gamma_0(N))$ with $f \neq 0$, we have $w_f^\pm \neq 0$.

**Proof.** Manin’s trick implies that the set of Manin symbols $\{u \cdot 0, u \cdot \infty\}_N$ for $u \in \Gamma_1(N)\backslash GL_2(\mathbb{Z})$ generates the first homology group of $X_1(N)$. Since the pairing (3.1) is non-degenerate, we finish the proof of the proposition. □

3.2. **Optimal periods.** In this section we discuss preliminary results to study the residual distribution of modular symbols.

Let $f$ be a newform of level $N$ and weight 2. Let $m$ be a maximal ideal of the Hecke algebra $\mathbb{T}_N$ such that the characteristic of $\mathbb{T}_N/m$ is $p$ and corresponds to $f$. Let $\mathbb{T}_{N,m}$ denote the completion of Hecke algebra $\mathbb{T}_N$ at $m$. There exists a Galois representation $\rho_m : Gal(\mathbb{Q}/\mathbb{Q}) \to GL_2(\mathbb{T}_N/m)$.

Let $C_1(N)$ be the set of cusps on $X_1(N)$. Consider $(X_1(N), C_1(N))$-relative homology sequence

$$0 \to H_1(X_1(N), \mathbb{Z}) \to H_1(X_1(N), C_1(N), \mathbb{Z}) \to H_0(C_1(N), \mathbb{Z}) \to \mathbb{Z} \to 0. \tag{3.3}$$

For a prime $q$ with $q \equiv 1 \pmod{Np}$, let $E_q = T_q - q(q) - 1$. The following is observed in Greenberg-Stevens [13]: The operator $E_q$ annihilates $H_0(C_1(N), \mathbb{Z})$ in (3.3). Since $E_q$ is a unit in $\mathbb{T}_{N,m}$ if $\rho_m$ is irreducible, we can conclude that $H_1(X_1(N), \mathbb{Z})_m$ is isomorphic to $H_1(X_1(N), C_1(N), \mathbb{Z})_m$. For a $\mathbb{Z}_p$-algebra $R$ with the trivial action of $\mathbb{T}_N$, we have a perfect pairing

$$H_1(X_1(N), R)_m \times H^1(\Gamma_1(N), R)_m \to R. \tag{3.4}$$

When $R$ is given by $\mathbb{C}$, the pairing is realized as the Poincaré pairing under the isomorphism $\mathbb{C}_p \simeq \mathbb{C}$.

Let $\mathcal{O}$ be an integral extension of $\mathbb{Z}_p$ including the Fourier coefficients of $f$. Assume $N \geq 3$, $p \nmid 2N$, and $\rho_m$ is irreducible. Then, there is a Hecke equivariant isomorphism

$$\delta^\pm : S_2(\Gamma_1(N), \mathcal{O})_m \simeq H^1(\Gamma_1(N), \mathcal{O})^\pm_m. \tag{3.5}$$

It is the isomorphism mentioned in Vatsal [39].

Let $\omega_f \in H_1(\Gamma_1(N), \mathbb{C})$ be a cohomology class corresponds to $f(z)dz$. Using the isomorphism (3.5) and the theorem of strong multiplicity one, the period $\Omega_f^\pm \in \mathbb{C}_p$ can be chosen (see Vatsal [39]) so that

$$\Omega_f^\pm \delta^\pm(f) = \omega_f \pm \omega_f. \tag{3.6}$$

Let us use this period with the same notation when we study the residual equidistribution of modular symbols. By previous discussion, one obtains $m_f^\pm(r) \in \mathcal{O}$ for
each $r$. It is known that for a newform $f_E$ corresponding elliptic curve $E$ over $\mathbb{Q}$, the period $\Omega_{f_E}^\pm$ can be chosen as the Néron periods $\Omega_E^\pm$ of $E$.

We write $y_f^\pm = (y_f^\pm) \in \mathcal{O}^{[\text{GL}_2(\mathbb{Z}) : \Gamma]}$, where

$$y_f^\pm := \{u \cdot 0, u \cdot \infty\} \cap \delta^\pm(f).$$

Note that we have

$$m_f^\pm(r) = \frac{(1 \pm 1)L(1, f)}{\Omega_f^\pm} + y_f^\pm \cdot \epsilon(r).$$

Then, we have the following non-vanishing result. Let $\pi$ be a uniformizer of $\mathcal{O}$.

**Proposition 3.2.** Assume $N \geq 3$, $p \nmid 2N$, and $\rho_m$ is irreducible. Let $f \not\equiv 0 \pmod{\pi}$. Then we have

$$y_f^\pm \not\equiv 0 \pmod{\pi}.$$

**Proof.** As before, the Manin symbols generate the first homology group of $X_1(N)$. Therefore from the perfectness of the pairing (3.4), the congruence $y_f^\pm \equiv 0 \pmod{\pi}$ implies that $\delta^\pm(f) \equiv 0 \pmod{\pi}$ by the perfectness of (3.4) with $R = \mathbb{F}_p$. However it is forbidden by the hypothesis using (3.5). In total, we finish the proof. □

### 3.3. Atkin-Lehner relation.

In order to deduce interval versions of the distribution results on $m_f^\pm$ in the next section, we verify the Atkin-Lehner relation for the modular symbols.

**Proposition 3.3.** For a cuspform $f$ for $\Gamma_0(N)$ of weight 2, we have

$$m_f^\pm = -m_f^{\ast\ast}$$

on the space $\Sigma_n$ with $N|n$, or on $\Omega_{M,\varphi N}$.

**Proof.** For $a \in (\mathbb{Z}/n\mathbb{Z})^\times$, let $u$ be the inverse of $a$ modulo $n$. Note that

$$\int_0^1 f(z)dz = \int_0^\infty f(z)\left(\begin{array}{cc}1 & a \\
 & 1 \end{array}\right) \left(\begin{array}{c}z \\
 \end{array}\right)dz.$$

Let $W_N$ be the Fricke involution. It can be easily seen that

$$\left(\begin{array}{cc}1 & -\frac{a}{n} \\
 \frac{a}{n} & 1 \end{array}\right) W_N^\ast \left(\begin{array}{c}1 \\
 0 \end{array}\right) = \left(1 \begin{array}{c} -\frac{a}{n} \\
 \frac{a}{n} \end{array}\right) \in \Gamma_0(N).$$

Hence the above integral can be rewritten as

$$\int_0^\infty f(z)\left(\begin{array}{cc}1 & \frac{a}{n} \\
 \frac{a}{n} & 1 \end{array}\right) \left(\begin{array}{c}z \\
 \end{array}\right)dz = -\int_0^\infty f(z)dz.$$

This concludes the proof of proposition. □

**Remark 3.4.** Using the argument in Kim-Sun [18], we can obtain the Atkin-Lehner relation even for general $n$. However, for a square-free $N$ and general $n$ we need to study the space $\Omega_{M,\varphi_n J,\varphi}$ where $g_n = N/(n, N)$, $\overline{g}_n$ is the inverse of $g_n$ modulo $N$, $\overline{g}_n J = \{r \in J \mid g_n r\}$. A problem is that $\overline{g}_n J$ is not an interval. This is the reason why we restrict ourselves to the space $\Sigma_n$ with $n$ being a multiple of $N$.  


3.4. Gaussian distribution of modular symbols. In this subsection, we show that the statements (1) and (2) of Conjecture B hold on average. By Theorem 2.9 and Corollary 2.13, we then prove the limit Gaussian distribution of modular symbols $m_f^\pm$ on $\Omega_{M,\psi}$.

Here we state the crucial probabilistic result due to Hwang, used in Baladi-Vallée [2]. This ensures the asymptotic normality of a random variable when its moment generating function is expressed as the quasi-power expression.

**Theorem 3.5** (Hwang’s Quasi-Power Theorem, Baladi-Vallée [2]). Assume that the moment generating functions for a sequence of functions $X_N$ on probability space $\Xi$ are analytic in a neighborhood $W$ of zero, and

$$\mathbb{E}[\exp(wX_N) | \Xi_N] = \exp(\beta_N U(w) + V(w)) (1 + O(\kappa_N^{-1}))$$

with $\beta_N, \kappa_N \to \infty$ as $N \to \infty$, $U(w), V(w)$ analytic on $W$, and $U''(0) \neq 0$. Then, the moments of $X_N$ satisfy

$$\mathbb{E}[X_N | \Xi_N] = \beta_N U'(0) + V'(0) + O(\kappa_N^{-1}),$$

$$\mathbb{V}[X_N | \Xi_N] = \beta_N U''(0) + V''(0) + O(\kappa_N^{-1})$$

$$\mathbb{E}[X_N^k | \Xi_N] = P_k(\beta_N) + O(\beta_N^{-k-1})$$

for some polynomials $P_k$ of degree at most $k \geq 3$. Furthermore, the distribution of $X_N$ on $\Xi$ is asymptotically Gaussian with speed of convergence $O(\kappa_N^{-1} + \beta_N^{-1/2})$, i.e.,

$$\mathbb{P}\left[ \frac{X_N - \beta_N U'(0)}{\sqrt{\beta_N}} \leq x \bigg| \Xi_N \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt + O\left(\frac{1}{\kappa_N + \beta_N^{1/2}}\right)$$

where the implicit constant is independent of $x$.

We may now state our results. We first prove an average version of Conjecture B.(1), and next claim (2) by making a particular choice of $\varphi$. Our proof is straightforward as this is a consequence of Corollary 2.13 with Theorem 3.5. Here let us recall that $\Omega_{\gamma} \otimes \varphi = \Omega_{M,\gamma,\varphi}$.

**Theorem 3.6.** Let $\Psi \in C^1(I_{\Gamma_0(N)} \cap \varphi) = I_{\gamma} \otimes \varphi$. Let $f$ be a newform for $\Gamma_0(N)$. Then, there exist a neighborhood $W$ of 0, a constant $\gamma > 0$, and analytic functions $U^\pm, B^\pm$ on $W$ such that $B^\pm$ is non-vanishing on $W$ and

$$\mathbb{E}[\exp(zm_f^\pm) | \Omega_{M,\psi}] = B^\pm(\gamma) M(U^\pm(z)) (1 + O(M^{-\gamma}))$$

with the $O$-term being uniform for $z \in W$. Hence we have:

1. The random variable $m_f^\pm$ on $\Omega_{M,\psi}$ follows asymptotic Gaussian distribution as $M$ goes to infinity.

2. There exist variance slope $C_f^\pm$ and shift $D_{f,\psi}^\pm$ such that $C_f^\pm$ is independent of $\Psi$ and

$$\mathbb{V}[m_f^\pm | \Omega_{M,\psi}] = C_f^\pm \log M + D_{f,\psi}^\pm + O(M^{-\gamma}).$$

3. Let $k \geq 3$. There exists a polynomial $Q_{\psi,k}$ of degree at most $k$ such that

$$\mathbb{E}[(m_f^\pm)^k | \Omega_{M,\psi}] = Q_{\psi,k} \log M + O((\log M)^{k-1} M^{-\gamma}).$$

**Proof.** Once we specialise $w = zw_f^\pm$ with $z$ near 0, all statements are immediate consequences of Theorem 2.9, Corollary 2.10 together with Proposition 3.1, Proposition 3.3, and Theorem 3.5. \qed
For a divisor \( d \) of \( N \), we shall denote \( \Omega_{M,1} \otimes \varphi_d \) briefly by \( \Omega_{M,d} \). Theorem 3.6 yields the limit Gaussian distribution of \( m^\pm_f \) on \( \Omega_{M,d} \). Again using Theorem 3.5, we can further deduce the following extra description about the variance of \( m^\pm_f \).

The constants can be expressed with various dynamical invariants which will be described in the proof of Proposition 2.2. More precisely, it is closely related to the partial derivatives of the dominant eigenvalue of transfer operator associated to an underlying dynamical model for modular partition vector. In particular, \( \frac{d}{dz}U^\pm(z)|_{z=0} \) admits an explicit expression with the entropy of the dynamical system.

It seems (Lhote [20]) that both of \( C^\pm_f \) and \( D^\pm_f,j \) are computable in polynomial time, but in general, there is no closed form under our approach. However, Petridis-Risager [30] successfully gave the exact formula for constants in terms of the special values \( L(1,\sym^2f) \) of symmetric square \( L \)-function when \( f \) is a newform.

3.5. Residual distribution of modular symbols. In this subsection, we show that for an elliptic newform \( f \), the \( p \)-integral random variable \( m^\pm_f \) on \( \Omega_{M,\varphi} \) and \( \Omega_{M,J,\varphi,N} \) is equidistributed mod \( p \). We present an evidence that the Gaussian distribution and residual distribution of \( m^\pm_E \) are not correlated.

**Theorem 3.7.** Assume that \( \overline{\rho}_{E,p} \) is irreducible and \( p \nmid N_E \).

1. For any integer \( a \) and \( c \geq 1 \), we have
   \[
   \mathbb{P} \left[ m^\pm_E \equiv a \pmod{p^c} \right] |_{\Omega} = p^{-c} + O(M^{-\gamma_1})
   \]
   for \( \Omega = \Omega_{M,J,N} \) or \( \Omega_{M,\varphi} \).

2. For \( z \) near 0, we have
   \[
   \mathbb{E}[\exp(zm^\pm_E)|\Omega_{M,\varphi} \cap (m^\pm_E \equiv a(p^c))] = \mathbb{E}[\exp(zm^\pm_E)|\Omega_{M,\varphi}](1 + O(M^{-\gamma_1}))
   \]

**Proof.** The desired statements follow from Corollary 2.17 and 2.19, Proposition 3.2, and Proposition 3.3. \( \square \)

We present a quantitative statement of non-vanishing mod \( p \) of special modular \( L \)-values.

**Corollary 3.8.** Assume that \( \overline{\rho}_{E,p} \) is irreducible and \( p \nmid N_E \). Then we have

\[
\# \left\{ \chi \in \overline{(\mathbb{Z}/n\mathbb{Z})}^\times \bigg| n \leq M, L_E(\chi) \neq 0(p^{1+\upsilon_p(\phi(n))}), \chi(-1) = \pm 1 \right\} \gg M.
\]

**Proof.** Let \( c \) be a number less than \( 1 - \sqrt{1 - \frac{2}{\pi^2}(1 - \frac{1}{p})} \). Let \( \overline{(\mathbb{Z}/n\mathbb{Z})}^\times \) be the set of Dirichlet characters modulo \( n \) that are even or odd according to the parity \( \pm \). Let us set

\[
T^\pm_M := \left\{ 1 < n \leq M \bigg| \exists \chi \in \overline{(\mathbb{Z}/n\mathbb{Z})}^\times, L_E(\chi) \neq 0(p^{1+\upsilon_p(\phi(n))}) \right\}.
\]

The statement follows once the following inequality is verified: \( \#T^\pm_M \geq cM \) for all sufficiently large \( M \). Let us assume the contrary, i.e., suppose that \( \#T^\pm_M < cM \) for infinitely many \( M \). Then for each \( n \notin T^\pm_M \) with \( 1 < n \leq M \) and \( m \in \overline{(\mathbb{Z}/n\mathbb{Z})}^\times \), we obtain

\[
\sum_{\chi \in \overline{(\mathbb{Z}/n\mathbb{Z})}^\times} \overline{\chi}(m)L_E(\chi) \equiv 0(p^{1+\upsilon_p(\phi(n))}).
\]
Then for all $r \in \Sigma_n$ with $n \not\in T_M^+$, we obtain $\hat{m}^r_{\phi}(r) \equiv 0 \pmod{p}$. From this, we can conclude

$$\sum_{1 < n \leq M} \phi(n) < \frac{1}{p} \sum_{1 < n \leq M} \phi(n), \text{ i.e., } \sum_{n \in T_M^+} \phi(n) > \left(1 - \frac{1}{p}\right) \sum_{1 < n \leq M} \phi(n).$$

Since $\#T_M^+ < cM$, the L.H.S. is smaller than or equal to

$$\sum_{M - cM < n \leq M} n \leq \frac{1}{2} (1 - (1 - c)^2)M^2.$$

Note that $\lim_{M \to \infty} \frac{1}{M^2} \sum_{n \leq M} \phi(n) = \frac{3}{π^2}$. Hence we obtain $\frac{1}{2} (1 - c)^2 \leq 1 - \frac{6}{π^2} (1 - \frac{1}{p})$, which is a contradiction to the choice of $c$. In total, we finish the proof. □

Remark 3.9. It seems that it is currently not doable to deduce an estimate on

$$\# \left\{ \chi \in (\mathbb{Z}/n\mathbb{Z})^\times \mid 1 < n \leq M, p \nmid \phi(n), L_E(\chi) \neq 0 (p), \chi(-1) = \pm 1 \right\}$$

from Corollary 3.8 or similar argument since the set $\{n \leq M \mid p \nmid \phi(n)\}$ is too thin as its size is asymptotic to $M(\log M)^{-1/(p-1)}$ (see Spearman-Williams [36]).

4. Transition to dynamics

The remaining part of this article will be devoted to explain in details how Proposition 2.2 can be proved. We give an underlying dynamical description for the modular partitions motivated by the work of Baladi-Vallée [2]. In this section, we establish that the Dirichlet series from modular partitions admits an explicit expression in terms of transfer operator associated to the skewed Gauss map.

4.1. Skewed Gauss dynamical system and transfer operator. Let $\Gamma$ be a subgroup in $\text{GL}_2(\mathbb{Z})$ of finite index $k = [\text{GL}_2(\mathbb{Z}) : \Gamma]$. Let us recall that the skewed Gauss map $T$ on $I_\Gamma = [0, 1] \times \Gamma \backslash \text{GL}_2(\mathbb{Z})$ is given by

$$T(x, v) := \left( T(x), v \begin{bmatrix} -m_1(x) & 1 \\ 1 & 0 \end{bmatrix} \right)$$

where $T$ denotes the usual Gauss map and $m_1(x)$ is the first digit of $x$. We call $(I_\Gamma, T)$ the skewed Gauss dynamical system. We shall consider the Gauss measure $\mu = \frac{dx}{\sigma_2(x)}$ on $[0, 1]$ and the counting measure $\nu$ on $\Gamma \backslash \text{GL}_2(\mathbb{Z})$. It is easily seen that $T$ is a measure-preserving transformation and in fact is ergodic with respect to $\mu \otimes \nu$. However, measure-theoretic properties will not be investigated in this article as we restrict our attention to its topological properties.

Observe that $I_\Gamma^2$ can be decomposed into a disjoint union

$$I_\Gamma^2 = \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k} \left[ \frac{1}{m+1}, \frac{1}{m} \right] \times \{u_j\}$$

with some fixed coset representatives $u_j$ for $\Gamma$ in $\text{GL}_2(\mathbb{Z})$. There is a denumerable subset $Q \subseteq \text{End}(I_\Gamma)$ that consists of inverse branches of $T$, that is,

$$Q = \{q_m \mid m \geq 1\}.$$
where an inverse branch \( q_m : I_G \rightarrow I_G \) is given by

\[
q_m(x, v) = \left( \frac{1}{m + x}, v \left[ \begin{array}{cc} 0 & 1 \\ 1 & m \end{array} \right] \right).
\]

Let \( F \subseteq Q \) be the final set that consists of branches corresponding to the final digits of continued fractions. In other words, it is given by

\[
F = \{ q_m \mid m \geq 2 \}.
\]

Let us denote by \( Q^{(n)} \) the set of inverse branches of the \( n \)-th iterate \( T^n \), which is equal to

\[
Q^{(n)} = Q \circ \cdots \circ Q = \{ q_{m_1} \circ q_{m_2} \circ \cdots \circ q_{m_n} \mid m_1, \ldots, m_n \geq 1 \}.
\]

Such \( n \) is called the depth of the inverse branches. Observe that setting \( x = 0 \) and \( v = I_2 \), we have

\[
q_{m_1} \circ q_{m_2} \circ \cdots \circ q_{m_n}(0, I_2) = \left( \frac{Q_{n-1}}{Q_n}, \left[ \begin{array}{cc} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{array} \right] \right)
\]

where \( P_n/Q_n = [0; m_1, \ldots, m_n] \). Note here that \( Q_{n-1}/Q_n = [0; m_n, m_{n-1}, \ldots, m_1] \).

For an inverse branch \( q = q_{m_1} \circ q_{m_2} \circ \cdots \circ q_{m_n} \) of depth \( n \) and \( i \leq n \), let us set the \( i \)-th part \( q^{(i)} \) of \( q \) as

\[
q^{(i)} := q_{m_1} \circ q_{m_2} \circ \cdots \circ q_{m_i} \in Q^{(i)}.
\]

Let us introduce the weighted transfer operator associated to the skewed Gauss system \((I_G, T)\). For two complex parameters \( s \in \mathbb{C} \) and \( w = (w_u) \in \mathbb{C}^{(\text{GL}_2(\mathbb{Z}))} \) and \( \Psi \in L^1(I_G) \), we define

\[
\mathcal{L}_{s, w} \Psi = \sum_{q \in Q} g_{s, w}(q) \cdot \Psi \circ q
\]

where \( g_{s, w} : Q \rightarrow \mathbb{C}^1(I_G) \) is a weight function given by

\[
g_{s, w}(q) := \exp(w \cdot 1 \circ \pi_2 q) \cdot J[n, q]^{s},
\]

the function \( I = (\mathbb{I}_u) \) is the vector-valued membership function on \( \Gamma \backslash \text{GL}_2(\mathbb{Z}) \), and \( J[n, q] \) denotes the Jacobian of the inverse branch \( \pi_1 q \) of \( T \). In a similar way, we can define a final operator \( F_{s, w} \) with respect to the final set \( F \). Using the expression (4.1), we can rewrite the operator in a more explicit way as

\[
\mathcal{L}_{s, w} \Psi(x, v) = \sum_{m \geq 1} \exp \left[ w \cdot \left[ \begin{array}{c} 0 \\ m + x \end{array} \right] \right] \Psi \left( \frac{1}{m + x}, v \left[ \begin{array}{cc} 0 & 1 \\ 1 & m \end{array} \right] \right).
\]

Note that this series representation converges absolutely for \( \Re(s) > \frac{1}{2} \) since the membership function \( I \) is bounded.

It is worthwhile to remark that a transfer operator is one of main tools for studying the statistical properties of trajectories of a dynamical system. Ruelle [33] first made a deep observation that the behavior of trajectories of dynamics can be well explained by spectral properties of the transfer operator.

We will see later that the main spectral properties of our transfer operator are established by a study on the properties of the set \( Q \) of inverse branches of \( T \) based on the following geometric properties of the Gauss dynamical system.

**Proposition 4.1** (Baladi-Vallée [2]). For any inverse branch \( h \) of the iterates \( T^n \) for \( n \geq 1 \):
(1) (Uniform contraction) There exists a constant \(0 < \rho < 1\) called a contraction ratio such that for any \(x \in [0, 1]\),
\[
|h'(x)| \ll \rho^n.
\]

(2) (Bounded distortion) There exists a distortion constant \(M > 0\) satisfying
\[
\frac{|h''(x)|}{|h'(x)|} \leq M
\]
for all \(x \in [0, 1]\).

(3) (Convergence on the half-plane) There is \(\sigma_0 < 1\) such that for all real \(\sigma > \sigma_0\),
\[
\sum_{m \geq 1} \sup_{x} |h'_m(x)|^\sigma < \infty.
\]

(4) The dual dynamical system \(([0, 1], T^*_s)\), whose set of inverse branches corresponding to the reverse composition
\[
h^*_s = h_{m_n} \circ h_{m_{n-1}} \circ \cdots \circ h_{m_1},
\]
of the branch \(h = h_{m_1} \circ h_{m_2} \circ \cdots \circ h_{m_n}\), again satisfies (1)–(3) and further \(h'(0) = h^*_s(0)\). Thus, the set of inverse branches of two dynamical systems are same.

The contraction ratio \(\rho\) is given by \(\limsup_n (\max |h'(x)|)^{1/n}\), which is equal to \(\frac{1}{2}\) in our case. This proposition directly implies that the operator \(L_{s,w}\) is well-defined as a perturbation of the weighted density transformer for \((\mathbb{R}s, \mathbb{R}w)\) near \((1, 0)\).

4.2. Key relation for Dirichlet series. In this subsection, we observe an underlying connection between the skewed Gauss dynamical system and modular symbols. In fact, the Dirichlet series \(L_\Psi(s, w)\) associated to modular partitions in §2.1 admits an explicit expression in terms of iterations of the transfer operator \(L_{s,w}\).

For each \(q \in \mathbb{Q}^{(n)}\), let us abuse the notation \(c\) to define a branch analogue of \(c(r)\) such that
\[
c(q) := \sum_{i=1}^n I \circ \pi_2 q^{(i)}.
\]
Then we set \(c(q) = (c_u(q))_u\) with \(c_u(q) = \sum_{i=1}^n I_u \circ \pi_2 q^{(i)}\). This can be interpreted that for all \((x, v) \in I_\Gamma\), we have
\[
c_u(q)(x, v) = \# \{1 \leq i \leq n | \pi_2 q^{(i)}(x, v) \in u\}.
\]
In order to represent \(L_\Psi(s, w)\) in terms of the iterates of the transfer operator, we first obtain expressions for the iterates as follows.

**Proposition 4.2.** Let \(\Psi \in L^\infty(I_\Gamma)\). Then for \(n \geq 0\) and \(\Re(s) > 1 + \frac{|x|}{2}\) we have
\[
F_{s,w} L^n_{s,w} \Psi = \sum_{q \in \mathcal{Q}(n) \circ F} \exp[w \cdot c(q)] J[\pi_1 q]^* \Psi \circ q.
\]

**Proof.** In order to apply the induction, we assume for \(n \geq 0\) that
\[
L^{n+1}_{s,w} \Psi = \sum_{q \in \mathbb{Q}(n)} \exp[w \cdot c(q)] J[\pi_1 q]^* L_{s,w} \Psi \circ q.
\]
By the definition of $L_{s,w}$, the last expression is equal to
\[
\sum_{m \geq 1} \sum_{q \in \mathbb{Q}^{(n)}} \exp[w \cdot \epsilon(q)] \exp[w \cdot \mathbf{I}(\pi_2 q_m \circ q)] J[\pi_1 q]^r \circ \Psi \circ q_m \circ q.
\]
It is easy to see that $\epsilon(q_m \circ q) = \epsilon(q) + \mathbf{I}(\pi_2 q_m \circ q)$ by the definition of $\epsilon$ and $J[\pi_1 q_m] \circ q] = J[\pi_1 q]^r \circ \Psi \circ q_m \circ q$ by the chain rule. This concludes the proof of proposition. \hfill \square

Let us set $\Omega := \mathbb{Q} \cap [0,1]$. There is an one-to-one correspondence between $\Omega$ and $\cup_{n \geq 1} \mathbb{Q}^{(n)} \circ \mathbf{F}(0, I_2)$ given by
\[
q : [0; m_1, \ldots, m_\ell] \mapsto q_{m_1} \circ \cdots \circ q_{m_\ell} (0, I_2)
\]
with $m_1, \ldots, m_{\ell-1} \geq 1$ and $m_\ell \geq 2$. Here we abuse the notation and use the same symbol $q$ to represent the correspondence (4.5). In particular, the rational number $r$ with $\ell(r) = n$ corresponds to the evaluation $q(r)$ of an inverse branch of depth $n$. For each $r \in \Omega$ it is easy to derive from the expression (4.3) the relation
\[
\epsilon(r) = \epsilon(q(r)).
\]
For $r \in \Omega$, let $Q(r)$ be the denominator of $r$. Since the definition of $L_q(s,w)$ converges absolutely for $\Re(s) > 1 + \frac{\log 2}{\log q}$ by Lemma 2.5, it can be rewritten as
\[
L_q(s,w) = \sum_{r \in \Omega} \frac{\Psi(r) \exp(w \cdot \epsilon(r))}{Q(r)^{2s}}.
\]
Specialising the two expressions of functions in (4.4) at $(0, I_2)$, we show:

**Proposition 4.3.** For $\Psi \in L_\infty(\Gamma)$ and $n > 0$, we obtain
\[
\mathcal{F}_{s,w} L_{s,w}^n \Psi(0, I_2) = \sum_{\ell(r) = n+1} \frac{\Psi(q(r)) \exp(w \cdot \epsilon(r))}{Q(r)^{2s}}
\]
where $\sum_{\ell(r) = n}$ means the sum over $r \in \Omega$ with $\ell(r) = n$.

**Proof.** For $r \in \Omega$, let us set $r = [0; m_1, \ldots, m_\ell]$ and $q(r) = (h_\ell(0), e(I_2))$ with $h = h_{m_1} \circ \cdots \circ h_{m_n}$. Observe from (4.2) that $h(0) = \frac{P_\ell}{Q_\ell}$ and $h_\ell(0) = \frac{Q_{\ell-1}}{Q_\ell}$. Moreover, note that $J[h_\ell(0)] = |h_\ell(0)|^2 = |h(0)|^2 = Q_\ell^{-2} = Q(r)^{-2}$ from Proposition 4.1.(4). Since $r \mapsto q(r)$ is an one-to-one correspondence, we obtain the statement by the identity (4.6). This finishes the proof. \hfill \square

Finally, previous discussion enables us to settle the following explicit expression for the Dirichlet series in terms of the transfer operator. Recall that $\mathcal{F} \Psi(x, v) = \Psi(1 - x, v)$ for a function $\Psi \in L_\infty(\Gamma)$.

**Theorem 4.4.** Let $\Psi \in L_\infty(\Gamma)$. Then we have
\[
L_{\Psi} (2s,w) = \mathcal{F}_{s,w} (I - L_{s,w}^0)^{-1} \Psi(0, I_2) + \mathcal{F}_{s,w} L_{s,w} (I - L_{s,w}^2)^{-1} \mathcal{F} \Psi(0, I_2).
\]

**Proof.** Let us use the same notations in the proof of Proposition 4.3. Notice that $r^* = \frac{Q_{\ell-1}}{Q_\ell}$ when $\ell = \ell(r)$ is odd and $r^* = 1 - \frac{Q_{\ell-1}}{Q_\ell}$ when $\ell$ is even. Hence, we can see that
\[
\Psi \left( r^*, \left[ \frac{P_{\ell-1}}{Q_{\ell-1}}, \frac{P_\ell}{Q_\ell} \right] \right) = \begin{cases} \Psi(h_\ell(0), e(I_2)) = \Psi(q(r)) & \text{if } \ell \text{ is odd} \\ \Psi(1 - h_\ell(0), e(I_2)) = \mathcal{F} \Psi(q(r)) & \text{if } \ell \text{ is even.} \end{cases}
\]
Then due to (4.7), the series $L_\Psi(s, w)$ for $\Re(s) \gg 1$ can be written as

$$
\sum_{\ell(r) \text{ even}} \frac{\Psi(q(r)) \exp(w \cdot c(r))}{Q(r)^{2s}} + \sum_{\ell(r) \text{ odd}} \frac{J\Psi(q(r)) \exp(w \cdot c(r))}{Q(r)^{2s}}
$$

From Proposition 4.3, we establish the desired expression. \qed

**Remark 4.5.** Under the assumption $J\Psi = \Psi$ or $\Psi^* = \Psi$, the above identity simply becomes

$$
L_\Psi(2s, w) = F_{s, w}(I - L_{s, w})^{-1}\Psi(0, I_2),
$$

which can be viewed as a natural extension of Baladi-Vallée [2, (2.17)]. Indeed, Theorem 4.4 reduces to their result by taking $\Psi = 1 \otimes 1$ and $\Gamma = \text{GL}_2(\mathbb{Z})$.

4.3. **Outline of dynamical discussion.** In view of Theorem 4.4, we will give a careful exposition of spectral analysis of the transfer operator in the remaining part of present article, which basically yields the crucial analytic properties of Dirichlet series described in Proposition 2.2.

We briefly outline the main ideas. The singularities and uniform polynomial bound of Dirichlet series in Proposition 2.2 are fully described by the characteristics of eigenvalues of the transfer operator $L_{s, w}$ and uniform estimate for the operator norm in a vertical strip. The starting point in §5 is to make a choice of the Banach space $C^1(I_\Gamma)$ on which the operator satisfies all desired properties. We observe that there is the unique dominant eigenvalue $\lambda_{s, w}$ and a spectral gap. This essentially follows from the geometric properties of Gauss map, which are given before in Proposition 4.1. On the other hand, the uniqueness and simplicity of $\lambda_{s, w}$ can be explained with the behavior of the skewed part of $T$.

In §6, we show the uniform polynomial bound of transfer operator, so-called the Dolgopyat-Baladi-Vallée estimate for the skewed Gauss dynamical system. Dolgopyat [11] established the uniform polynomial estimate for the iterates of a transfer operator associated to a certain dynamical system with finite Markov partitions satisfying Uniform Non-Integrability (UNI) condition. Baladi-Vallée [2] modified the ideas to the Gauss dynamical system with countably infinite partitions. Our point is to observe that the estimate can be controlled only by the behavior of the Gauss map, so we simply modify their arguments.

5. **Spectral analysis of transfer operator**

In this section, we present a dynamical analysis on the transfer operator associated to the skewed Gauss dynamical system. Let $\Gamma$ be a subgroup of $\text{GL}_2(\mathbb{Z})$ of finite index and later assumed to be a congruence subgroup $\Gamma_0(N)$ in §5.3. For the remaining part of the article, we write $s := \sigma + it \in \mathbb{C}$. Let us also recall $w = x + iy \in C^{[\text{GL}_2(\mathbb{Z})] \cdot \Gamma}$.

The spectral properties of transfer operator strongly depend on a function space on which the operator acts. We study the spectrum of $L_{s, w}$ acting on the Banach space of the continuously differentiable functions on $I_\Gamma$. Indeed, for $(s, w)$ with $(\sigma, x)$ near $(1, \mathbf{0})$, we show that $L_{s, w}$ is quasi-compact. The operator thus has a positive eigenvalue of maximum modulus which is unique and algebraically simple, and there is a spectral gap between this dominant eigenvalue and the remaining part of the spectrum.
5.1. Function space and basic properties. In this subsection, we collect some basic properties of transfer operator.

Let us consider the space of the continuously differentiable functions
\[
C^1(I_\Gamma) = \{ \Psi : I_\Gamma \to \mathbb{C} \mid \Psi \text{ and } \partial \Psi \text{ are continuous} \}
\]
where the derivative \( \partial \) on \( C^1(I_\Gamma) \) is defined by the partial derivative with respect to the first coordinate
\[
\partial \Psi := \frac{\partial}{\partial x} \Psi.
\]
Since \( \Gamma \) is of finite index in \( \text{GL}_2(\mathbb{Z}) \), the space \( C^1(I_\Gamma) \) is just \( \bigoplus_{\Gamma \backslash \text{GL}_2(\mathbb{Z})} C^1([0,1]) \) and its elements are the linear combinations of tensor type map \((f \otimes g)(x, v) := f(x)g(v)\).

It is easily seen that this is a Banach space with the norm \( ||\Psi||_1 := ||\Psi||_0 + ||\partial \Psi||_0 \).

Recall that \( ||\Psi||_0 \) denotes the supremum norm. We deal with the transfer operator \( \mathcal{L}_{s,w} \) acting on \( C^1(I_\Gamma) \). One can notice that the action is well-defined for \( s \) and \( w \) with \( \sigma > \frac{1}{2} \) directly due to Proposition 4.1. The following shows that the operator further acts boundedly.

Proposition 5.1. For real \( \sigma_0 > \frac{1}{2} \), there exists a constant \( M_{\sigma_0} > 0 \) such that for all \( \sigma > \sigma_0 \), we have

1. \( ||\mathcal{L}_{s,w} \Psi||_0 \leq M_{\sigma_0} ||\Psi||_1 \) and
2. \( ||\partial \mathcal{L}_{s,w} \Psi||_0 \leq M_{\sigma_0}(1 + |s|)||\Psi||_1 \).

Proof. For \( q_m = (h_m, e_m) \in Q \), Proposition 4.1 shows that for all \( \sigma > \sigma_0 \), \( x \) near \( 0 \) and \( (x, v) \in I_\Gamma \), the infinite series
\[
\sum_{m \geq 1} \exp[w \cdot I \circ e_m(v)] \cdot |h_m'(x)|^s
\]
is not identically zero and converges since \( |h_m'(x)| = O(m^{-2}) \). We may write it as \( M_{\sigma_1} \), then
\[
\sup_{(x,v) \in I_\Gamma} |\mathcal{L}_{s,w} \Psi(x,v)| \leq M_{\sigma_1} |\Psi \circ q_m(x,v)|
\]
\[
\leq M_{\sigma_1} \sup_{(x,v) \in I_\Gamma} |\Psi(x,v)|.
\]

Also, we notice
\[
\sup_{(x,v) \in I_\Gamma} ||\partial \mathcal{L}_{s,w} \Psi(x,v)|| \leq \sum_{m \geq 1} \exp[w \cdot I \circ e_m(v)] \left| \frac{-s}{(m + x)^{2s+1}} \cdot \Psi(h_m(x), e_m(v)) \right|
\]
\[
+ \sum_{m \geq 1} \exp[w \cdot I \circ e_m(v)] \left| \frac{1}{(m + x)^{2s}} \cdot \partial \Psi(h_m(x), e_m(v)) \right|
\]
The last expression is less than or equal to
\[
M_{\sigma_2} ||\Psi||_0 + M_{\sigma_1} ||\partial \Psi||_0 \leq M_{\sigma_0}(1 + |s|)||\Psi||_1
\]
by taking \( M_{\sigma_0} := \max(M_{\sigma_1}, M_{\sigma_2}) \). Thus, we obtain the inequalities. \( \square \)

An important remark is the following: The operator depends analytically on \((s, w)\) for \((\sigma, x)\) near \((1, 0)\). By perturbation theory, we can conclude that properties of \( \mathcal{L}_{\sigma,x} \) extend to \( \mathcal{L}_{s,w} \) for such complex \( s \) and \( w \).
Proposition 5.2. For $s$ and $w$ with $(\sigma, x)$ near $(1, 0)$, the map $(s, w) \mapsto \mathcal{L}_{s, w}$ is analytic.

Proof. Let $F_{C^1, \Psi}$ be an element of a bidual of $C^1(I_T)$ defined by $F_{C^1, \Psi}(L) := L(\Psi)$. Then by the Hahn-Banach theorem, it suffices to show that $F_{C^1, \Psi}(\mathcal{L}_{s, w})$ depends holomorphically on $s$ and $w$. Note that the operator $\mathcal{L}_{s, w}$ is the sum of the component operators, each of which is a composition of the form

$$\Psi \mapsto \mathcal{L}_{s, w, [m]} \Psi := \exp(w \cdot 1_{\sigma_2} q_m) J[\pi_1 q_m]^s \cdot \Psi \circ q_m.$$  

It is easy to see that $F_{C^1, \Psi}(\mathcal{L}_{s, w, [m]})$ is holomorphic in $s$ and $w$, respectively and therefore we finish the proof since the series $\sum_{m \geq 1} \left| \frac{s \log(m + x)}{(m + x)^{s+1}} \right|$ converges for $s$ with $\sigma > \frac{1}{2}$. \hfill $\square$

5.2. Dominant eigenvalue and spectral gap. In this subsection, we describe the spectrum of transfer operator $\mathcal{L}_{s, w}$ acting on $C^1(I_T)$. More precisely, for real pair $(\sigma, x)$ near $(1, 0)$, the operator $\mathcal{L}_{\sigma, x}$ admits a real positive eigenvalue $\lambda_{\sigma, x}$ of maximum modulus and there is a spectral gap, i.e., the essential spectral radius is strictly less than $\lambda_{\sigma, x}$. This basically follows from the quasi-compactness of the transfer operator. We shall remark that the operator $\mathcal{L}_{s, w}$ does not act compactly on $C^1(I_T)$. Let us begin by stating the following sufficient condition for quasi-compactness due to Hennion.

Theorem 5.3 (Hennion [16]). Let $\mathcal{L}$ be a bounded operator on a Banach space $X$, endowed with two norms $||.||$ and $||.||'$ satisfying $\mathcal{L}(\{ \phi \in X : ||\phi|| \leq 1 \})$ is conditionally compact in $(X, ||.||')$. Suppose that there exist two sequences of real numbers $r_n$ and $t_n$ such that for any $n \geq 1$ and $\phi \in X$, one has the inequalities

$$(5.1) \quad ||\mathcal{L}^n\phi|| \leq t_n||\phi||' + r_n||\phi||.$$  

Then the essential spectral radius of $\mathcal{L}$ is strictly less than $\liminf_{n \to \infty} r_n^{1/n}$.

Inequalities of the form (5.1) are often called the Lasota-Yorke type in the theory of dynamical system. Let us remark that the inequality is extremely useful for our purpose: This enables us not only to show the quasi-compactness of $\mathcal{L}_{s, w}$ on $C^1(I_T)$, but also to obtain an explicit estimate for iterates $\mathcal{L}_{s, w}^n$. Using this, we will prove the uniform spectral bound for transfer operator in §6.

Proposition 5.4. For $s$ and $w$ with $(\sigma, x)$ near $(1, 0)$, there exists a constant $C > 0$ such that for any $n \geq 1$ and $\Psi \in C^1(I_T)$

$$||\partial \mathcal{L}_{s, w}^n \Psi||_0 \leq C(||s||||\Psi||_0 + \rho^n||\partial \Psi||_0).$$

Proof. We proceed by induction. For $n = 1$, it is clear from Proposition 5.1 and Proposition 4.1 (1). For $n > 1$, we shall consider $\mathcal{L}_{s, w} = \mathcal{L}_{s, w}^{n-1}$, then we obtain

$$\sup_{(x, v) \in I_T} |\partial \mathcal{L}_{s, w}^n \Psi(x, v)|$$

$$\leq \sum_{m_1 \geq 1} \exp[w \cdot 1_{\sigma_1}(v)] \left| \frac{s}{(m_1 + x)^{2s+1}} \right| \mathcal{L}_{s, w}^{n-1} \Psi(h_{m_1}(x), e_{m_1}(v))$$

$$+ \sum_{m_2 \geq 1} \exp[w \cdot 1_{\sigma_1}(v)] \left| \frac{1}{(m_2 + x)^{2s}} \right| \partial \mathcal{L}_{s, w}^{n-1} \Psi(h_{m_2}(x), e_{m_2}(v)).$$
The last expression is less than or equal to
\[ C_0|s||\Psi||_0 + C_1 \rho^{n-1}||\partial \Psi||_0 \leq C(|s||\Psi||_0 + \rho^n||\partial \Psi||_0) \]
for suitable constants \( C_0, C_1, \) and \( C \) by the induction hypothesis. \( \square \)

Note that the embedding of \((C^1(I_\Gamma), ||.||_1)\) into \((C^1(I_\Gamma), ||.||_0)\) is a compact operator, since \( \Gamma \) is of finite index in \( GL_2(\mathbb{Z}) \). Hence by Theorem 5.3 and Proposition 5.4, we have the quasi-compactness of \( L_{s,w} \) on \( C^1(I_\Gamma) \). Thus, we obtain:

**Proposition 5.5.** For \((\sigma, x)\) in a compact neighborhood \( K \) of \((1,0)\):

1. The operator \( L_{\sigma,x} \) has a positive real eigenvalue \( \lambda_{\sigma,x} \) of maximal modulus. The corresponding eigenfunction \( \Phi_{\sigma,x} \) is also positive, and the associated eigenmeasure \( \mu_{\sigma,x} \) of its adjoint operator is a positive Radon measure. In particular, \( \lambda_{1,0} = 1 \).
2. There is a spectral gap, i.e., the essential spectral radius is strictly bounded by the dominant eigenvalue \( \lambda_{\sigma,x} \).

**Proof.** We mainly refer to Baladi [3] for a general theory. Let us first recall that \( L_{\sigma,x} \) is defined with a positive real-valued weight function \( g_{\sigma,x} : Q \to C^1(I_\Gamma) \) satisfying \( g_{\sigma,x}(q) = \exp(x \cdot \pi_2 q) \cdot J[\sigma x q]^T \). By the inequality from Proposition 5.4, we see

\[
||L^\sigma_{\sigma,x} \Psi||_1 \leq ||\Psi||_1 \cdot ||L^\sigma_{\sigma,x} 1||_0
\]

where \( 1 := 1 \otimes 1 \) denotes the constant function. Let \( \lambda_{\sigma,x} \) be the limit of \( ||L^\sigma_{\sigma,x} 1||^{1/n}_0 \). Then we have \( \lambda_{\sigma,x} > 0 \) since \( g_{\sigma,x} \) is positive and the spectral radius formula and inequality (5.2) yield that \( R(L_{\sigma,x}) \) is smaller than \( \lambda_{\sigma,x} \). We also have

\[
\lim_{n \to \infty} \sup_{||\Psi||_1 = 1} ||L^\sigma_{\sigma,x} \Psi||^{1/n}_1 \geq \lim_{n \to \infty} ||L^\sigma_{\sigma,x} 1||^{1/n}_1 \geq \lim_{n \to \infty} ||L^\sigma_{\sigma,x} 1||^{1/n}_0 ,
\]

which enables us to obtain the converse inequality. It follows that \( R(L_{\sigma,x}) \) is equal to \( \lambda_{\sigma,x} \).

It is a standard argument in functional analysis to show that \( \lambda_{\sigma,x} \) is an eigenvalue of \( L_{\sigma,x} \) with the desired properties of the eigenfunction \( \Phi_{\sigma,x} \). In fact, it is a straightforward adaption of Baladi [3, §1.5], Baladi-Vallée [2, Proposition 0], or Ruelle [33, §4.9]. A spectral gap in (2) immediately follows from the quasi-compactness due to Proposition 5.4. \( \square \)

**Remark 5.6.** By Proposition 5.2 with analytic perturbation theory on the dominant part of spectrum, Proposition 5.5(2) extend to \( L_{s,w} \) for the complex pair \((s,w)\) close to \((1,0)\).

We shall remark that the following will play a central role in the proof of Proposition 2.2: the analyticity guarantees the existence of the map \( s_0 \) on the neighborhood \( W \) of \( 0 \) by implicit function theorem.

**Proposition 5.7.** If \((s,w)\) is a complex pair near \((1,0)\), then \( \lambda_{s,w} \), \( \Phi_{s,w} \), and \( \partial \Phi_{s,w} \) are well-defined and analytic.

Let us note that Proposition 5.5 further yields an additional description about the spectrum of \( L_{s,w} \) for \((s,w)\) away from \((1,0)\).

**Proposition 5.8.** Assume that \((\sigma, x)\) is near \((1,0)\). For a fixed \( v > 0 \) and \( y \) with \( v < |y| \leq \pi \), the eigenvalues of \( L_{s,w} \) are away from 1.
Proof. It can be shown with no difficulty that if there exists \((\sigma,x)\) near \((1,0)\) such that \(\lambda_{\sigma,x}\) belongs to the spectrum of \(\mathcal{L}_{s+it,x+y}\) for any \((t,y) \neq (0,0)\), then there is \(\Psi \in C^1(I_{\Gamma})\) with \(|\Psi| = 1\) such that for any \(q = (h,v) \in \mathbb{Q}^n\), we have

\[
\exp \left[ \Psi \left( x \right) \right] = \Psi \left( x \right) \quad \text{for all } (x,v) \in I_{\Gamma} \quad \text{(e.g., see Baladi-Vallée [2, Lemma 7]).}
\]

Suppose that 1 is an eigenvalue of \(\mathcal{L}_{1+it,w}\). One can easily observe from Theorem 6.1.(a) below that \(t\) must be equal to 0 and thus for any \(q = (h,v) \in \mathbb{Q}^n\), we have

\[
\exp \left[ \Psi \left( x \right) \right] = \Psi \left( x \right) \quad \text{for all } (x,v) \in I_{\Gamma} \quad \text{(e.g., see Baladi-Vallée [2, Lemma 7]).}
\]

Along with a spectral gap from Proposition 5.5, this yields that 1 does not belong to the spectrum of \(\mathcal{L}_{1+it,w}\) for \(t \neq 0\) and \(v < |y| \leq \pi\). Finally by perturbation theory, we have a compact neighborhood \(K_1\) contained in \(K\) from Proposition 5.5 and \(d > 0\) such that for \((s,w)\) with \((\sigma,x) \in K_1\) and \(v < |y| \leq \pi\), the distance between 1 and the spectrum of \(\mathcal{L}_{s,w}\) is at least \(d\). \(\square\)

### 5.3. Characteristics of dominant eigenvalue.

In this subsection, we show that the dominant eigenvalue \(\lambda_{\sigma,x}\) is algebraically simple and there are no other eigenvalues on the spectral radius in particular for the congruence subgroup \(\Gamma = \Gamma_0(N)\). This will be explained crucially by the interaction of the skewed components of \(T\) while the existence of dominant eigenvalue and a spectral gap are mainly due to the generic properties of the Gauss map \(T\).

Let us recall that Manin-Marcolli [22] had earlier considered the skew-product Gauss map which is almost the same as ours and studied the eigenvalues of maximum modulus of a certain associated transfer operator acting on the space of holomorphic functions with an invariant cone. More precisely, they established that the operator acts compactly and the dominant eigenvalue is simple and unique. Here, we shortly observe this from the another perspective.

For a dynamical system \((X,f)\), we call the map \(f\) topologically mixing if for any non-empty open subsets \(U\) and \(V\) in \(X\), there exists a positive integer \(N\) such that \(f^n(U) \cap V \neq \emptyset\) for all \(n \geq N\). We then remark the following classical Ruelle-Perron-Frobenius-type result under our setting:

**Theorem 5.9.** Let the assumptions be as in the above. Then, the topological mixing of the map \(T\) implies that the dominant eigenvalue \(\lambda_{\sigma,x}\) is simple and further there are no eigenvalues of maximum modulus except itself.

**Proof.** Since the skew-product \(T\) is a compact extension of the Gauss map and the associated operator \(\mathcal{L}_{\sigma,x}\) is defined with a positive weight function \(g_{\sigma,x}\) (recall §4.1), this can be obtained directly by following the standard arguments in Baladi [3, Theorem 1.5 (4)(5)], together with Proposition 4.1 and Proposition 5.5. See also Baladi-Vallée [2] and Parry-Pollicott [32] for more details. \(\square\)

Let us first observe the following. Fix the coset representatives for \(\Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z})\), then:

**Proposition 5.10.** For any \(u \in \Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z})\) and \(\ell \geq 3\), we have

\[
\left\{ u \left[ \begin{array}{cc} -m_1 & 1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{cc} -m_2 & 1 \\ 1 & 0 \end{array} \right] \cdots \left[ \begin{array}{cc} -m_\ell & 1 \\ 1 & 0 \end{array} \right] \mid m_1,\ldots,m_\ell \geq 1 \right\} = \Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z}).
\]
Proof. Due to Shimura [34], we have an explicit set of coset representatives for \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{Z}) \) which can be determined by

\[
\mathcal{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \mid (c, d) = 1, \ d|N, 0 < c \leq \frac{N}{d} \right\}.
\]

Therefore, we shall choose \( \mathcal{R}^\pm := \mathcal{R} \cup \mathcal{R} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \) for a set of coset representatives for \( \Gamma_0(N) \) in \( \text{GL}_2(\mathbb{Z}) \). In this proof, by \([c : d]\), we denote \([\frac{a}{d} \ 1]\) \(\in \mathcal{R}^\pm \).

Multiplication by \( \begin{bmatrix} 1 & \frac{a}{d} \\ 0 & 1 \end{bmatrix} \) shows

\[
\begin{bmatrix} c & d \\ \frac{-m}{c} & 1 \end{bmatrix} \begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d - mc & c \\ \frac{d}{c} & \frac{c}{d} \end{bmatrix}.
\]

Thus by the congruence condition, first for \( d = N \), the element \([1 : 0]\) goes to the case \( d = 1 \), in other words, we get \([0 : 1]\), \([1,1]\), \ldots , \([n-1 : 1]\) after one step. Then note that \([0 : 1]\) only maps to \([1 : 0]\) and for the other \([c : 1]\), we obtain all \([c : d]\) with \( d = 1 \) and \( N \) when \((c, N) = 1 \) and with \( d > 1 \) when \((c, N) \neq 1 \). For \( d > 1 \), the element goes to the case \( d = N \) and also \([c : d]\) again with \( d > 1 \).

Hence for each \( u \in \Gamma_0(N) \setminus \text{GL}_2(\mathbb{Z}) \), after the third iterations of right multiplication, it becomes spread everywhere, i.e., one obtains the all elements. \( \square \)

Proposition 5.10 suggests us to observe the mixing property of \( T \) for \( \Gamma = \Gamma_0(N) \) and hence by Theorem 5.9, we conclude the described statements about the dominant eigenvalue.

Proposition 5.11. The skewed Gauss map \( T \) is topologically mixing.

Proof. Recall that \( I_{\Gamma} \) can be decomposed into a disjoint union \( \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k} I_m \times \{u_i\} \), where \( I_m := \left[ \frac{1}{m+1} \frac{1}{m} \right] \), \( k = |\text{GL}_2(\mathbb{Z}) : \Gamma| \) and \( u_i \) are the fixed coset representatives. Take any non-empty open sets \( U \) and \( V \) in \( I_{\Gamma} \). Then, one can assume that \( U \) is of the form \((a, b) \times \{u\}\) for some \( 0 < a < b < 1 \) and \( u \in \Gamma \setminus \text{GL}_2(\mathbb{Z}) \). Since \((a, b)\) is non-empty and the Gauss map \( T \) satisfies the strong markov property, i.e., \( T(I_m) = [0,1] \) for all \( m \geq 1 \), we have \( T^n(a, b) = [0, 1] \) for \( n \geq N \) and some positive integer \( N \).

Once we have the full image on the first coordinate, by Proposition 5.10, we obtain the all elements in the skewed component at least three more iterations as well. Hence for \( \ell \geq 3 \) and \( n \geq N + \ell \), we have \( T^n(U) \cap V = I_{\Gamma} \cap V \neq \emptyset \). \( \square \)

6. Uniform Dolgopyat bound in a vertical strip

In this section, we obtain the Dolgopyat-Baladi-Vallée estimates for our transfer operator. This states that for \( s \) away from the real axis, the operator \( L_{s, w} \) satisfies a uniform polynomial growth with respect to the norm \( ||\cdot||_{(t)} \) on \( C^1(I_{\Gamma}) \). Recall that it is given by \( ||\Psi||_{(t)} = ||\Psi||_0 + \frac{1}{|t|} ||\partial_t \Psi||_0 \) for \( t \neq 0 \). We remark that this norm is equivalent to \( ||\cdot||_{(1)} \). Consequently, along with the results from §5, we finally present a proof of Proposition 2.2 at the end of this section.

6.1. Uniform polynomial decay. In this subsection, we obtain the uniform polynomial bound for the operator norm of transfer operator. Dolgopyat [11] first established the result of this type for the plain transfer operators associated to certain dynamical systems with a finite set of inverse branches, which depends on one complex parameter \( s \). Let us roughly overview his ideas for the proof:
Due to the spectral properties of transfer operator, the main estimate can be reduced to $L^2$-norm estimate, which is decomposed into a sum of oscillatory integrals over the pairs of the inverse branches. The sum is divided into two parts.

Relatively separated pairs of inverse branches consist in one part, in which the oscillatory integrals can be simply dealt with the Van der Corput Lemma.

In order to control the other part that consists of close pairs, the dynamical system must satisfy the Uniform Non-Integrability (UNI) condition, which explains that there are a few such pairs.

This groundbreaking work has been generalised to possible applications by Parry-Pollicott [31], Naud [27], and recently Oh-Winter [29] mostly for counting problems for closed geodesics or periodic orbits of a map on compact manifolds. In particular, Baladi-Vallée [2] modified the UNI condition to obtain Dolgopyat estimate for the weighted transfer operators associated to the Gauss dynamical system with a countably many inverse branches. An outline of their proof, hence ours, goes almost similarly as above. We observe that the desired main $|||_{(t)}$-estimate for our transfer operator is basically controlled by the behavior of inverse branches of the Gauss map.

Let us begin by introducing the UNI property for the Gauss dynamical system established by Baladi-Vallée [2]. For any $n \geq 1$ and for two inverse branches $h$ and $k$ of $T^n$, the distance is defined by

$$
\Delta(h, k) = \inf_{x \in [0, 1]} |\psi'_{h,k}(x)|
$$

where $\psi_{h,k}$ is a map given by

$$
\psi_{h,k}(x) := \log \left| \frac{h'(x)}{k'(x)} \right|
$$

They obtained:

**Proposition 6.1** (Baladi-Vallée [2, Lemma 6]). *The Gauss dynamical system $([0, 1], T)$ satisfies the UNI condition, i.e., it satisfies the properties:

(a) Let $m$ be the Lebesgue measure on $[0, 1]$. For $0 < a < 1$, one obtains

$$
m \left( \bigcup_{\Delta(h, k) \leq \rho^n} k([0, 1]) \right) \ll \rho^n
$$

where $\rho$ denotes the contraction ratio.

(b) For any $h$ and $k$, one has $\sup_{x \in [0, 1]} |\psi'_{h,k}(x)| < \infty$.

Notice that both statements essentially follow from Proposition 4.1. The condition (a) states that there are very few pairs of inverse branches for which the distance $\Delta(h, k)$ is small.

We claim that the spectral gap from Proposition 5.5 allows us to reduce our desired estimate to $L^2$-norm estimate and then observe that Proposition 6.1 is sufficient to control the $L^2$-estimate. Let us now explain in detail. We consider the normalised operator $\tilde{L}_{s,w}$ which is defined by

$$
\tilde{L}_{s,w} \Phi := \lambda_{s,x}^{-1} \Phi^{s} \cdot \Phi_{s,x} \cdot \Psi
$$
with the dominant eigenvalue \( \lambda_{\sigma,x} \) and the corresponding eigenfunction \( \Phi_{\sigma,x} \). One can immediately see that \( \tilde{L}_{\sigma,x} \) on \( C^1(I_G) \) has a spectral radius 1 and fixes the constant function 1, i.e., \( \tilde{L}_{\sigma,x} \mathbf{1} = \mathbf{1} \). Note that the adjoint operator \( \tilde{L}_{\sigma,x}^\ast \) also fixes the corresponding eigenmeasure. A direct computation shows that the normalised operator again satisfies the Lasota-Yorke inequality as follows:

**Proposition 6.2.** For \((s, w)\) with \((\sigma, x)\) in a real neighborhood \( K \) of \((1, 0)\) and all \( n \geq 1 \), there exists a constant \( A_K > 0 \)

\[
||\tilde{L}_{s,w}^n \Psi||_1 \leq A_K (||s|| ||\Psi||_0 + \rho^n ||\partial \Psi||_0).
\]

**Proof.** Observe that

\[
\tilde{L}_{s,w}^n \Psi(x, v) = \lambda_{\sigma,x}^{-n} \Phi_{\sigma,x}^{-1}(x, v) \sum_{(h,e) \in Q(n)} \exp \left[ \sum_{i=1}^{n} w \cdot e^{(i)}(v) \right] \times |h'(x)|^{s} (\Phi_{\sigma,x} \cdot \Psi)(h(x), e(v))
\]

\[
\leq ||\Psi||_0 \cdot |\tilde{L}_{s,w}^n \mathbf{1}(x, v)|.
\]

Hence we obtain \( ||\tilde{L}_{s,w}^n \Psi||_0 \leq ||\Psi||_0 \). Differentiating (6.1) gives

\[
|\partial \tilde{L}_{s,w}^n \Psi| \leq |\partial (\Phi_{\sigma,x}^{-1}) \tilde{L}_{s,w}^n \Psi|
\]

\[
+ \lambda_{\sigma,x}^{-n} \Phi_{\sigma,x}^{-1} \sum_{(h,e) \in Q(n)} \exp \left[ \sum_{i=1}^{n} w \cdot e^{(i)} \right] \partial (|h'|^{s} \cdot \Phi_{\sigma,x} \Psi \circ (h, e)).
\]

By Proposition 5.7, the first term of the last expression can be controlled by the perturbation

\[
|\partial \Phi_{\sigma,x}^{-1}| \leq A_K \cdot \frac{1}{\Phi_{\sigma,x}}
\]

for a suitable \( A_K > 0 \) depending on a compact neighborhood \( K \) of \((1, 0)\). The second term can be written as a sum of two parts and each satisfies that \( |s||h''| \cdot ||h'|^{s-1}|| = |s| \frac{h''}{|h'|^{s-1}} \leq |s|M|h'|^{s} \) and

\[
|h'||\partial (\Phi_{\sigma,x} \cdot \Psi) \circ (h, e) | \leq \rho^n (||\partial \Phi_{\sigma,x} \cdot \Psi||_0 + ||\Phi_{\sigma,x} \cdot \partial \Psi||_0)
\]

by the bounded distortion and uniform contraction property from Proposition 4.1. Hence, we obtain the statement. \( \square \)

**Lemma 6.3.** For \((\sigma, x)\) in \( K \) and for all \( n \geq 1 \)

\[
||\tilde{L}_{s,w}^n \Psi||_0^2 \ll A_{\sigma,x}^2 ||\tilde{L}_{1,0}^n \Psi||_0^2
\]

for some \( A_{\sigma,x} = \lambda_{\sigma,x}^{-1} \sqrt{\lambda_{2\sigma-1,2x}} > 0 \) depending only on \( K \).
Proof. By the Cauchy-Schwarz inequality, the normalised value $|\tilde{L}_{\sigma,x}^n \Psi|^2$ equals to

$$\lambda_{v,x}^{-n} \Phi_{\sigma,x}^{-1} \cdot L_{\sigma,x}^n \left( \Phi_{\sigma,x} \cdot \Psi \right)^2$$

$$\leq \lambda_{v,x}^{-2n} \Phi_{\sigma,x}^{-2} \left( \sum_{(h,e) \in Q^{(n)}} \exp \left[ \sum_{i=1}^n 2x \cdot I \circ \epsilon(i) \right] \right)^2$$

$$\leq \lambda_{v,x}^{-2n} \left( \sum_{(h,e) \in Q^{(n)}} \exp \left[ \sum_{i=1}^n 2x \cdot I \circ \epsilon(i) \right] \right)^2 \left( \sum_{(h,e) \in Q^{(n)}} |h'|^{2\sigma-1} \right).$$

Notice that the second factor is simply bounded by

$$\sum_{(h,e) \in Q^{(n)}} |h'| \cdot |\Psi|^2 \circ (h,e) = L_{1,0}^n |\Psi|^2 \ll \tilde{L}_{1,0}^n |\Psi|^2$$

and the first factor can be estimated as

$$\sum_{(h,e) \in Q^{(n)}} \exp \left[ \sum_{i=1}^n 2x \cdot I \circ \epsilon(i) \right] \ll \sum_{(h,e) \in Q^{(n)}} |h'|^{2\sigma-1} \ll \lambda_{v,x}^{-2n} \cdot 2^{\sigma-1}.$$
Theorem 6.5. For any $0 < \xi < \frac{1}{5}$ and $(s, w)$ with $(\sigma, x)$ in $K$, $n \geq 1$, and for $|t| \geq \frac{1}{\varphi^*}$, we have
\[
\|L^n_{s, w}\|_{1(t)} \ll \eta^n \cdot \lambda_{\sigma, x} |t|^\xi
\]
for a constant $0 < \eta < 1$.

Proof. Set $n_0 = n_0(t) := \lfloor \alpha \log |t| \rfloor$. Then for $n_1 = n_1(t) \geq n_0$, we have
\[
\|\tilde{L}^{n_0}_{s, w} \Psi\|_0^2 \ll \|\tilde{L}^{n_1 - n_0}_{s, w} [\tilde{L}^{n_0}_{s, w} \Psi]\|_0^2
\ll A_{\sigma, x}^{2(n_1 - n_0)} \|\tilde{L}^{n_1 - n_0}_{s, w} \Psi\|_0^2
\]
by Lemma 6.3. Further by the normalisation and Proposition 5.5, we get
\[
\tilde{L}^{n_0}_{s, w} [\tilde{L}^{n_0}_{s, w} \Psi] = \int_{\Gamma} |\tilde{L}^{n_0}_{s, w} |^2 + O(R_1^{n_1 - n_0}) \|\tilde{L}^{n_0}_{s, w} \Psi\|_1
\]
where $R_1 < 1$ denotes the subdominant spectral radius of $\tilde{L}_{1, 0}$. By Proposition 6.4, we can write
\[
\|\tilde{L}^{n_1}_{s, w} \Psi\|_0^2 \ll A_{\sigma, x}^{2(n_1 - n_0)} \left( \int_{\Gamma} |\tilde{L}^{n_0}_{s, w} \Psi|_1^2 + R_1^{n_1 - n_0} |t| \|\Psi\|_1^2 \right)
\ll A_{\sigma, x}^{2(n_1 - n_0)} \left( \frac{1}{|t|^\beta} + R_1^{n_1 - n_0} |t| \right) \|\Psi\|_1^2.
\]
Taking $n_1$ to be sufficiently large enough to satisfy $R_1^{n_1 - n_0} |t| = O(|t|^{-\beta})$ as well as $A_{\sigma, x}^{n_1 - n_0} = O(|t|^{-\beta/2})$, we get
\[
\|\tilde{L}^{n_1}_{s, w} \Psi\|_0 \ll \frac{\|\Psi\|_1}{|t|^{\beta}}.
\]
Repeated application of Lasota-Yorke inequality from Proposition 6.2 enables us to write
\[
\|\tilde{L}^{n_1}_{s, w} \Psi\|_1 \ll \|s\| \|\tilde{L}^{n_1}_{s, w} \Psi\|_0 + \rho^{n_1} \|\tilde{L}^{n_1}_{s, w} \Psi\|_1 \ll |t| \frac{\|\Psi\|_1}{|t|^{\beta}}.
\]
and accordingly we get $$||\mathcal{E}^2_{s,w}||_{(t)} \ll \frac{1}{|t|^3}$$. Writing any integer \( n = (2n_1)s + r \) with \( r < 2n_1 \), we finally obtain

$$||\mathcal{E}^n_{s,w}||_{(t)} \leq ||\mathcal{E}^r_{s,w}||_{(t)} ||\mathcal{E}^{2n_1}_{s,w}||_{(t)} \ll \frac{1}{|t|^3},$$

which directly leads to the assertion after a choice of \( \xi \).

6.2. **Proof of Proposition 2.2.** We conclude the discussion by presenting the proof of Proposition 2.2. In view of Theorem 4.4, the main spectral properties of transfer operator \( \mathcal{L}_{s,w} \) describe the analytic behavior of the Dirichlet series \( L_\Psi(2s,w) \) in a vertical strip.

First we find a representation of the residue operator of the quasi-inverse.

**Proposition 6.6.** Let \( \mathcal{R}_w \) be the residue operator of \( (I - \mathcal{L}_{s,w})^{-1} \) at \( s_0 = s_0(w) \). Then when \( w \) is near \( 0 \), we have for all \( \Psi \in C^1(I_F) \) that

$$\mathcal{R}_w \Psi = \Phi_{s_0,w} \int_{I_F} \Psi \Phi^{-1}_{s_0,w} d\mu_{s_0,w}.$$

**Proof.** By Remark 5.6, we have \( \mathcal{L}^n_{s,w} = \lambda^n_{s,w} \mathcal{P}_{s,w} + N^n_{s,w} \) and

$$\mathcal{P}_{s,w} \Psi = \lim_{n \to \infty} \frac{1}{\lambda^n_{s,w}} \mathcal{L}^n_{s,w} \Psi = \Phi_{s,w} \int_{I_F} \Psi \Phi^{-1}_{s,w} d\mu_{s,w}$$

for \( w \) near \( 0 \). Since \( \mathcal{P}_{s_0(w),w} \) is the residue operator of \( (1 - \lambda_{s,w})^{-1} \mathcal{P}_{s,w} \), we obtain the statement.

Recall that the residue of \( L_\Psi(s,w) \) at \( s_0 = s_0(w) \) is \( E_\Psi(w) \). Hence we obtain statements on the residue of the Dirichlet series:

**Proposition 6.7.** When \( w \) is near \( 0 \), then for all \( \Psi_1, \Psi_2 \in C^1(I_F) \) we have

(6.3)

$$|E_{\Psi_1}(w) - E_{\Psi_2}(w)| \ll ||\Psi_1 - \Psi_2||_{L^1}.$$

Moreover, we also have

$$E_\Psi(0) = \frac{1}{2 \log 2} \int_{I_F} \Psi(x,v) dxdv.$$

**Proof.** By Theorem 4.4, the residue is

$$\mathcal{F}_{s_0,w}(I + \mathcal{L}_{s_0,w})^{-1} \mathcal{R}_w \Psi(0, I_2) + \mathcal{F}_{s_0,w} \mathcal{L}_{s_0,w}(I + \mathcal{L}_{s_0,w})^{-1} \mathcal{R}_w \mathcal{F}(0, I_2),$$

which is equal to

$$\frac{\mathcal{F}_{s_0,w} \Phi_{s_0,w}(0, I_2)}{1 + \lambda_{s_0,w}} \left( \int_{I_F} \Phi^{-1}_{s_0,w} d\mu_{s_0,w} + \lambda_{s_0,w} \int_{I_F} (\mathcal{F} \Psi) \Phi^{-1}_{s_0,w} d\mu_{s_0,w} \right).$$

From this, we obtain the first statement.

For the second one, observe that \( \int_{I_F} \Phi^{-1}_{1,0} d\mu_{1,0} = \int_{I_F} \Psi(x,v) dxdv \) and

$$\int_{I_F} \Psi \Phi^{-1}_{s_0,w} d\mu_{s_0,w} = \int_{I_F} (\mathcal{F} \Psi) \Phi^{-1}_{s_0,w} d\mu_{s_0,w}.$$

Note also that

$$\mathcal{F}_{s,w} \Phi_{s,w}(0, I_2) = \lambda_{s,w} \Phi_{s,w}(0, I_2) - \exp(w_u) \Phi_{s,w}(1, u)$$

where \( u = \Gamma_0(N)[0,1] \). Hence, we have

$$\mathcal{F}_{1,0} \Phi_{1,0}(0, I_2) = \Phi_{1,0}(0, I_2) - \Phi_{1,0}(1, u) = \frac{1}{2 \log 2}.$$
This proves the second statement and we finish the proof of the proposition. □

We are ready to give:

Proof of Proposition 2.2. We proceed similarly as Baladi-Vallée [2]. Let us recall that the membership function $I$ is defined to satisfy, for $q = (h, e) \in Q$, $\exp(w \cdot I \circ \pi_2 q) = \exp(w u)$ with a single variable $w u \in C$. Thus, the transfer operator can be simply written as

$$
\mathcal{L}_{s, w} \Psi(x, v) = \sum_{(h, e) \in Q} \exp(w u) |h'(x)|^s \cdot \Psi(\pi_2 q, e(v))
$$

where $e(v) \in u$ for some $u \in \Gamma_0(N) \backslash \text{GL}_2(\mathbb{Z})$. From this, we see $\lambda_{1,0} = 1$ with the corresponding eigenfunction $\Psi_{1, 0} = \psi_1 \otimes 1$, where $\psi_1$ denotes the Gauss density $\psi_1(x) = \frac{1}{\log 2} \frac{1}{1+x}$.

Moreover, by Proposition 5.2 with a simple modification of the result in Broise [7] on log convexity, one can observe that the derivative of the map $(s, w) \mapsto \lambda_{s, w}$ at $(1, 0)$ satisfies $\lambda'_{s, w}(1, 0) \neq 0$ and the Jacobians $[\lambda'_{s, w}(1, 0)]$ and $[\lambda''_{s, w}(1, 0)]$ are invertible. Thus by implicit function theorem, we get the map $s_0$ from the neighborhood $W$ of 0 to $C$ such that $\lambda_{s_0(s, w)} = 1$ and the Hessian of $s_0$ is non-singular at $w = 0$.

For the statements (2)–(7), we observe that by Remark 5.6, a spectral decomposition with the unique simple dominant eigenvalue $\lambda_{s, w}$ extends to $L_{s, w}$ for $(s, w)$ in a complex neighborhood of $(1, 0)$: $L_{s, w} = \lambda_{s, w} P_{s, w} + N_{s, w}$, where $P_{s, w}$ denotes the projection to the dominant eigenspace $E_{\lambda_{s, w}}$ and the spectral radius of $N_{s, w}$ is strictly smaller than the modulus of $\lambda_{s, w}$. More generally, for all $n \geq 1$, we have

$$
L_{s, w}^n = \lambda_{s, w}^n P_{s, w} + N_{s, w}^n.
$$

Along with Theorem 4.4, Theorem 5.5, Proposition 5.8, Proposition 5.11, and Theorem 6.5, the rest of the proof then directly follows in the same lines as in Baladi-Vallée [2, Lemma 8 and 9]. Finally, the statement (3) follows from Proposition 6.6 and 6.7. □

References

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