# **Mahler measure and special values of $L$-function: Deninger's cohomological method**

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(joint work with Wei He)

The Mahler measure is defined, for a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ with complex coefficients, by

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(x_1, \cdots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \tag{1}$$

where $T^n$ denotes the real $n$-torus.

The quantity $m(P)$ naturally arises from the diverse contexts in number theory or dynamical system, for instance as a height function on polynomials or a topological entropy. Here, we focus on its mysterious connection to special values of $L$-functions.

One of the first relations between Mahler measure and special $L$-values goes back to Smyth and Boyd, who discovered

$$m(1 + x + y) = L'(-1, \chi_3),$$
$$m(1 + x + 1/x + y + 1/y) = r_E \cdot L'(0, E),$$

where $\chi_3 : (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$ is the unique non-trivial Dirichlet character modulo 3, $E$ is an elliptic curve of conductor 15 defined by the given equation and $r_E \in \mathbb{Q}^\times$. There have been numerous such instances, which precisely take the form:

$$\frac{L'(0)}{m(P)} \in \mathbb{Q}^\times \tag{2}$$

where $P \in \mathbb{Q}[x_1^{\pm}, x_2^{\pm}]$, and either $L(s-1, \chi)$ or $L(s) = L(s, E)$. See [5, 2].

Naturally this lead to several questions, e.g. given a Laurent polynomial $P$, can we determine which $L$-function is related to $m(P)$? Conversely, given a Dirichlet character or an elliptic curve, is the special $L$-value always related to the Mahler measure of one or more polynomials? In general, these questions are out of reach.

In this note, we discuss the breakthrough work of Deninger, which partially and theoretically confirms an answer in the case of elliptic curves. We then describe our result that is a generalisation of Deninger's cohomological method for Dirichlet characters.

## 1. Deninger's method

We briefly outline the work of Deninger, later further refined by the work of Besser–Deninger.

In [3], Deninger showed that if $P$ does not vanish on $T^n$, then the Mahler measure of $P$ can be viewed as a Deligne period of the motive $Z_P$ associated to the variety defined by $P = 0$. This implied that the value is related to the Beilinson regulator map, hence to $L$-values.

Let $P \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ be a non-zero Laurent polynomial and set $P^*$ to be the leading coefficient of $P$ as a polynomial in $x_n$. Let $Z_P \subset \mathbb{G}_{m,\mathbb{C}}^n$ be the zero locus of $P$ and $A_P := \{|z_1| = \cdots = |z_{n-1}| = 1, |z_n| \leq 1\}$ be the subspace.

- Suppose $P^* \neq 0$ on $T^{n-1}$, applying Jensen's formula, we have
$$m(P) = m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{A_P} \log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$
Denote by $\eta_P$ the smooth differential form in the integration.
- Reinterpret the integration as a period pairing of $\eta_P$ and $A_P$. To do so, one has to consider $Z_P^{\mathrm{reg}} \subset Z_P$ on which $\eta_P$ is a closed form and $A_P$ is compactly embedded. Then the integral becomes
$$\langle r_D(\{P, x_1, \cdots, x_n\}), [A_P] \otimes (2\pi i)^{1-n} \rangle$$
where $r_D$ denotes the Beilinson regulator applied to a motivic cohomology class induced by $P$ and $\log|x_i|$. Abusing the notation, denote by $r_D$ this pairing in the below.

This suggests that given $P$ satisfying certain non-vanishing assumptions, the Mahler measure is given by the regulator map, accordingly related to certain $L$-function associated to the motive defined by $P$. Conversely, given an arithmetic object, one can try to find a corresponding polynomial in this cohomological context. Indeed, in [1], they proved that there exists $P_E \in \mathbb{Q}[x_1, x_2]$ that corresponds $E_{/\mathbb{Q}}$ with CM by imaginary quadratic fields of class number 1 and with technical assumptions such that

$$m(P_E) - m(P_E^*) = r_E \cdot L'(0, E), \quad r_E \in \mathbb{Q}^\times. \tag{3}$$

## 2. Result

We now state our main result, which is a $\mathrm{GL}_1$-analogue of (3) for Dirichlet $L$-values. The proof rests on Deninger's foundational work described in Section 1 with an appropriate choice of the polynomial and motive attached.

Let $H_M^*$ and $H_D^*$ denote the Motivic and Deligne cohomology group respectively.

**Theorem 1** (He.-L [4], 2025). *There exists $\Psi := \Psi_N \in \mathbb{Z}[x_1, x_2]$ associated to $N$-th cyclotomic polynomials such that the regulator map*

$$r_D : H_M^2(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2)) \to H_D^2(Z_\Psi, \partial A_\Psi, \mathbb{R}(2)) \xrightarrow{\frac{1}{2\pi i}\int_{A_P}} \mathbb{C}$$

*is given by*

$$r_D(\{\Psi, x_1, x_2\}) = m(\Psi) - m(\Psi^*) = \sum_{\chi \in \widehat{G}} r_\chi \cdot L'(-\varepsilon, \chi)$$

*where $G = \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, $\varepsilon = 0$ when $\chi$ is even and $\varepsilon = 1$ when $\chi$ is odd.*

Then we have a natural short exact sequence

$$0 \to H_M^1(X_\Psi, \mathbb{Q}(2)) \to H_M^2(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2))^{i=-1} \to H_M^1(Y_\Psi, \mathbb{Q}(1)) \to 0 \tag{4}$$

induced from the long exact sequence of relative motivic cohomologies for some suitable integral models $X_\Psi$ and $Y_\Psi$ of $\mathrm{Spec}(\mathbb{Q}(\mu_N))$, where $i$ is induced by the involution on $(Z_\Psi, \partial A_P)$ from the interchange of two variables.

**Theorem 2** (He.-L [4], 2025)**.** *Under certain $\mathbb{Q}$-linear independence assumption on partial $L$-values at $0$ and $-1$, we have a canonical splitting*

$$H^2_M(Z_\Psi, \partial A_\Psi, \mathbb{Q}(2))^{i=-1} \cong H^1_M(X_\Psi, \mathbb{Q}(2)) \oplus H^1_M(Y_\Psi, \mathbb{Q}(1))$$

*of the exact sequence* (4) *that is compatible with the regulator maps.*

*Thus we have an induced $G$-module structure on $H^2_M$ and for each $\chi \in \widehat{G}$, we have $r_D(\{\Psi, x_1, x_2\}^\chi) = r_\chi \cdot L'(-\varepsilon, \chi)$.*

*Concluding remarks.*

In Theorem 1, we remark that it is given by a linear combination of $L$-values in contrast to the Besser–Deninger formula. This appearance is essentially due to the Galois conjugate structure in our $\mathrm{GL}_1$ setting.

In Theorem 2, we show that one can extract the single $L$-values by splitting the exact sequence under extra $\mathbb{Q}$-linear independent assumption. Hence it also strongly suggests an answer to the converse question; Given a Dirichlet character $\chi$ modulo $N$, it is likely to have some $\Psi^\chi_N \in L[x_1, x_2]$ such that $m(\Psi^\chi)$ is related to $L'(-1, \chi)$, where $L$ is a number field containing the Hecke field $\mathbb{Q}(\chi)$.

(Date: September 10, 2025)

## References

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